Mathematical Reviews

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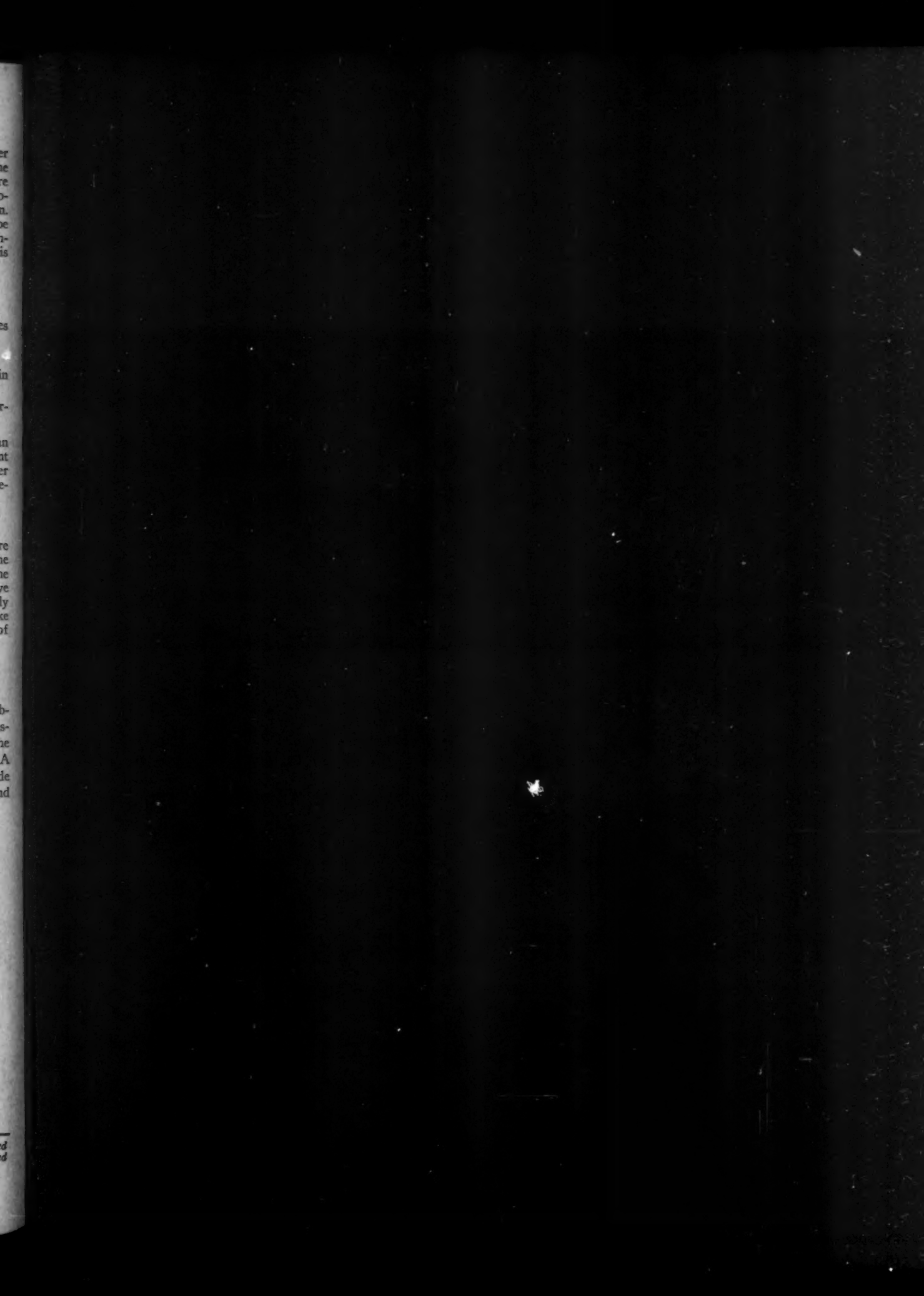

Mathematical Reviews

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FOUNDATIONS

*Quine, Willard Van Orman. **Elementary Logic.** Ginn and Company, Boston, 1941. vi+170 pp. $2.25.

This is an elementary text on logic, intended, presumably, for beginning students in philosophy. For that purpose it represents a radical departure from tradition. It is strictly a book on modern symbolic logic conceived in the author's own way. The subjects covered are the algebra of propositions and the first order calculus of predicates (quantification theory). These are developed not as abstract mathematical systems but as interpreted symbolic procedures for expressing "the basic logical constructions and reasonings involved in ordinary discourse." The meaning of the author's fundamental operations is thus fixed; and the possibility of other interpretations is not even mentioned. About half of the book is devoted to a detailed analysis of translations from ordinary language into logical symbols; in this the author makes a real contribution. The other half is concerned with explaining the techniques for transforming expressions into their equivalents. This is based on considerations akin to the truth table method, rather than on axioms and rules in the usual sense. The book has the disadvantage that it uses a peculiar terminology; such expressions as "stencil," "extrication," "frame" and "compositional transformation" are as unintelligible to the expert in logic (if unfamiliar with Quine) as they are to the beginner. Again, the author expresses everything explicitly in terms of conjunction, denial and the existential quantifier, whereas many processes would be more perspicuous if other connectives were introduced by definitions. (For example, the process of reduction to conjunctive normal form is here replaced by a process which, to a mathematician at least, is far less intelligible.) Thus the book shows a pronounced bias towards conceptions which are peculiar to the author. It is, however, a clearly written, scholarly and original book; within its limitations it is a noteworthy addition to the expository literature of the subject. *H. B. Curry.*

*Church, Alonzo. **The Calculi of Lambda-Conversion.** Annals of Mathematics Studies, no. 6. Princeton University Press, Princeton, N. J., 1941. ii+77 pp. $1.25.

This is a systematic and detailed exposition of certain researches of the author and his students. Essentially it is a revision of the author's Princeton lectures of 1936 [reviewed in J. Symbolic Logic **2**, 39–40 (1937)]. This revision is a thorough-going one; practically everything has been rewritten. Some of the principal changes are as follows. The introductory portions have been amplified, although some of the concrete examples and all of the exercises have been omitted. The notation has been improved, in that the author now uses exclusively the Schönfinkel notation (whereby the application of f to a is simply fa). A proof of the Church-Rosser theorem [Trans. Amer. Math. Soc. **39**, 472–482 (1936)] has been included. Altogether the statement of the fundamentals of the original calculus of λ-conversion is much more complete and explicit. In regard to the formulation of arithmetic, that subject is also treated more fully; some account is given of the results of Kleene and Turing with references to the original papers for details. A decided improvement is made in the treatment of the Gödel enumeration, where the earlier complicated theory of "metads" is abandoned in favor of a simpler theory. The theory of λ-δ-conversion is postponed to the last chapter where its principal properties are stated briefly by analogy; this chapter also contains some account of the theory of λ-K conversion which is ignored in the earlier work. On the other hand the author's interesting theory of quantification is omitted entirely, the treatise ending with a sketch of the existential quantifier without negation. There is a full bibliography; and references are made to this bibliography from the text for the sources of the more important results. The monograph is a well organized presentation of these interesting researches, and is valuable to anyone with a serious interest in them. It would be a useful reference book for the specialist if only it were provided with an adequate index. *H. B. Curry* (State College, Pa.).

Curry, Haskell B. The paradox of Kleene and Rosser. Trans. Amer. Math. Soc. **50**, 454–516 (1941). [MF 5720]

The author calls attention to the proof of Kleene and Rosser [cf. in particular Ann. of Math. (2) **36**, 630–636 (1935)] that certain systems of formal logic are inconsistent in the sense that every formula which can be expressed in their notation is also demonstrable. The author seeks to simplify and clarify the presentation of this important paradox whose original complete discussion ran through a whole series of papers, and which has been called intricate and complicated. The author starts by examining the paradox in "a vague preliminary way" explaining the central idea in intuitive terms. This informal survey reads as follows: "One of the goals toward which mathematicians strive in setting up formal systems is completeness—by which I mean not completeness in the technical sense, but simply the adequacy of the system for some purpose or other. There are two kinds of such completeness which especially concern us in this paper; both of them are desirable properties of formal systems of mathematical logic. I shall call these combinatorial completeness and deductive completeness, respectively. They may be roughly explained as follows. A theory is combinatorially complete if and only if every expression $\mathfrak{M}$ formed from the terms of the system and an auxiliary indeterminate or variable x can be represented within the system as a function of x (that is, we can form in the system a function whose value for any argument is the same as the result of substituting that argument for x in $\mathfrak{M}$). A theory is deductively complete if whenever we can derive a proposition B on the hypothesis that another proposition A holds then we can derive without hypothesis a third proposition (such as $A \supset B$) expressing this deducibility. Combinatorial completeness is thus a property relating to the possible constructions of terms (or formulas)

within the system; deductive completeness relates to the possible derivations. Deductive completeness is a well-known property of certain systems; whereas combinatorial completeness has only been achieved in recent years." "The essence of the Kleene-Rosser theorem is that it shows that these two kinds of completeness are incompatible, that is, that any system which possesses both of them is inconsistent. The argument is essentially a refinement of the Richard paradox; it shows, in fact, that the Richard paradox can be set up formally within the system."

An added footnote calls attention to a simpler discussion to be contained in a paper "now in preparation" (Aug., 1941): The inconsistency of certain formal logics.

A. A. Bennett (Providence, R. I.).

Rosser, Barkley. The independence of Quine's axioms *200 and *201. J. Symbolic Logic 6, 96–97 (1941).

The axioms "*200" and "*201" are those so designated in Quine's book [Mathematical Logic, W. W. Norton, New York, 1940; cf. these Rev. 2, 65]. The author proves the independence of *200. For *201 the author shows that if Quine's system is consistent then *201 is independent, thus establishing its "contingent independence."

A. A. Bennett (Providence, R. I.).

Rosser, Barkley. On the many-valued logics. Amer. J. Phys. 9, 207–212 (1941). [MF 6040]

Since new viewpoints in physics, such as the indeterminacy principle, seem to call for a new kind of logic with more flexibility and a more complex structure than the older logics and, since such a new logic, to be applicable to physics, should be able to serve as a basis for defining numbers, the author is led to consider the sort of system obtained by adjoining the membership relation and the universal quantifier to a Post-Łukasiewicz many-valued logic. Arithmetic may then be developed along the lines of W. V. Quine's Mathematical Logic [W. W. Norton and Co., New York, 1940; these Rev. 2, 65]. The system which results is not worked out in detail in the present paper. Some of the difficulties are discussed, such as the multiplicity of possible definitions of conjunction and negation in many-valued logics, and the resulting multiplicity of kinds of integers when the arithmetic has been defined in terms of logical operations. The possibility of using the many-valuedness of the logic to avoid some of the paradoxes is considered in connection with the Russell paradox, but the Burali-Forti paradox is not discussed. *O. Frink.*

Müller, Hans Robert. Algebraischer Aussagenkalkül. Akad. Wiss. Wien, S.-B. IIa. **149**, 77–115 (1940). [MF 4936]

This paper is based on the idea of representing operations on several variables in an $n+1$ valued logic by means of polynomials in the complex field which assume only the truth values $0, \epsilon, \epsilon^2, \cdots, \epsilon^n=1$, when the variables are restricted to the same set of values. Here ϵ is a primitive nth root of unity, and 1 and 0 represent truth and falsity. When the author actually works with these polynomials, however, in the case of the 2 and 3 valued logics (the only cases treated in detail), he considers them to be polynomials in the field of integers reduced modulo 2 or 3.

By way of illustration, a number of problems of familiar type are worked out for the 2 valued logic in this notation. These include cases of the decision problem, the independence and consistency of axiom systems, and conditions that logical operations obey certain formal laws, such as the postulates for a lattice, or the associative and distributive laws. These examples can be thought of also as illustrating the use of the symmetric difference operation to replace logical addition in Boolean algebra, as suggested by M. H. Stone and others.

It is claimed that this polynomial representation of operations makes easier the comparison of different many-valued logics, such as those of Post and Łukasiewicz-Tarski, and that it also is useful in the problem of deciding which operations of a many valued logic are to be considered as generalizations of the familiar operations of the 2-valued logic, a question recently treated by J. B. Rosser [cf. the preceding review]. *O. Frink* (State College, Pa.).

Pankajam, S. On the formal structure of the propositional calculus. I. J. Indian Math. Soc. (N.S.) **5**, 49–61 (1941). [MF 5696]

The author considers a calculus of propositions which constitutes a distributive lattice but not necessarily a Boolean algebra. Here the operations of addition and of multiplication are each associative and commutative; each is distributive with respect to the other. Finally $a+(a\cdot b)$ is equivalent to a. By non-formal reasoning the author concludes the existence of 0 and 1. Four laws concerning negation are discussed and their relative entailment examined, namely: (N_1) law of contradiction: $aa'=0$; (N_3) if $ax=0$ then $x<a$; (N_3) law of excluded middle: $a+a'=1$; (N_4) law of double negation: $(a')'=a$. The discussion involves results of M. H. Stone, A. Heyting, A. Tarski, O. Frink and others. *A. A. Bennett* (Providence, R. I.).

Tarski, Alfred. On the calculus of relations. J. Symbolic Logic 6, 73–89 (1941).

The aim of this paper is to awaken interest in the Peirce-Schröder calculus of binary relations, and to call attention to some of its outstanding problems. Two methods of setting up this calculus as a formal logical system are described. In the author's first formal system the calculus of relations is included as part of a more comprehensive logical theory called the elementary theory of relations. In this system there are variables for individuals as well as variables for relations, all the usual logical constants, including the quantifiers, and eleven strictly relational constants. The latter include symbols for particular relations, such as identity, diversity and the universal relation, as well as other symbols for operations on relations, such as converse, complement and relative product. Sentences of the system are constructed by means of the logical connectives from elementary sentences of the form xRy and $R=S$. In addition to the usual axioms and rules of inference of the calculus of functions, there are twelve axioms involving the relational constants which amount to definitions of these constants.

The theorems of this more comprehensive theory which contain no individual variables (and hence no quantifiers) constitute the calculus of relations proper. In the second, more restricted theory, quantifiers and individual variables are omitted from the start. The axioms which previously served to define the relational constants are replaced by others containing no individual variables. These new axioms are shown to be theorems of the first system. Proofs of 17 theorems of the second theory are outlined. In particular it is shown that by making use of the relative product

operation, the negation of an equation can be replaced by an equation. It is pointed out that elementary group theory is contained in that part of the calculus which deals with one-to-one relations.

Some metalogical questions discussed are the following. Are the two theories equivalent if in the first theory we are restricted to those sentences which contain no individual variables? Is every model of one of these systems isomorphic to a concrete system of binary relations, that is, representable as a system of classes of ordered pairs of individuals? These questions remain open. It is stated that two further metalogical problems, the decision problem for the calculus of relations, and the question of whether every sentence of the first theory is equivalent to one of the second, will be answered in the negative in another paper. It is made clear that the calculus of relations, though outwardly similar to the calculus of classes, is much richer than the latter, and, though lacking quantifiers, still presents many of the difficulties and peculiarities of the calculus of functions.

O. Frink (State College, Pa.).

Wernick, William. Complete sets of logical functions. Trans. Amer. Math. Soc. **51**, 117–132 (1942). [MF 6091]

The author calls a set of functions of two variables in the two-valued logical calculus complete if all functions of the calculus can be formally defined in terms of the functions of the set, and no proper subset has this property. He shows that there are just two complete sets of single functions, nine complete sets of two functions each, exactly six complete sets of three functions and no complete sets of more than three functions. In deriving these results he makes use of the concept of the field of functions generated by a given set A. This consists of all functions (of two variables) formally definable in terms of the functions of A, and is similar to a notion used by E. L. Post [Amer. J. Math. **43**, 163–185 (1921); Two-Valued Iterative Systems of Mathematical Logic, Princeton University Press, Princeton, N. J., 1941; these Rev. **2**, 337]. *O. Frink.*

Dotterer, Ray H. A generalization of the antilogism. J. Symbolic Logic **6**, 90–95 (1941).

The original Ladd-Franklin definition of an antilogism as a set of three propositions of the forms $ab=0$ or $ab\neq 0$, the denial of any one of which is implied by the conjunction of the others, is generalized by removing the restriction to three propositions, and allowing also propositions of the form $a\neq 0$. It is proved that a set of propositions of the forms considered is a generalized antilogism if the following conditions hold: (1) Exactly one of the propositions is an inequation. (2) There are just as many terms as there are propositions containing two terms. (3) Each term appears in at least two propositions, and is positive in one and negative in another unless it occurs in the inequation, when it is consistently positive (or negative) throughout. These conditions provide a convenient test for the validity of non-syllogistic modes of inference with any number of premises, such as sorites or immediate inference, by determining whether the set of propositions obtained by denying the conclusion contains a generalized antilogism. In particular it is shown that the method is applicable to the so-called disputed moods of the traditional syllogism, when these are rendered valid by adding a third premise of form $a\neq 0$.

O. Frink (State College, Pa.).

Giorgi, Giovanni. Riflessioni sui fondamenti primi della teoria degli insiemi. Pont. Acad. Sci. Acta **5**, 35–40 (1941). [MF 5885]

The author discusses the notion of existence and related notions in relation to axiomatic set theory. This is no novelty. *H. B. Curry* (State College, Pa.).

Hosiasson, Janina Lindenbaum. Induction et analogie: comparaison de leur fondement. Mind **50**, 351–365 (1941). [MF 6076]

This article seeks to compare, from the point of view of their theoretical foundation, two species of reasoning, (1) inductive and (2) by analogy. The author's chief conclusion is that induction is well-founded in an almost general manner, while reasoning by analogy is well-founded only under special conditions. In either case the inquiry concerns the increase in conviction (or degree of belief or subjective attitude) attaching to a thesis (hypothesis or law) constituting a logical reason. The author considers a numerical function $\mathfrak{C}(a, b)$ symbolizing the certitude of a with respect to b where "a" and "b" are variable names of certain theses. An arithmetic for the certitude function paraphrasing that for probabilities is developed from certain propositions "evidently applicable." Upon manipulation the postulates are shown to yield the following: If (a) $\mathfrak{C}(h, s)>0$ and (b) $\mathfrak{C}(f_1, s)<1$, then $\mathfrak{C}(h, f_1 s)>\mathfrak{C}(h, s)$, where h is the hypothesis, f_1 an established fact and s designates data in our possession before the establishment of f_1. Hence inductive reasoning is well-founded. However under like conditions reasoning by analogy in its most general form is not well founded. More than one form of sufficient condition for the latter is provided. *A. A. Bennett* (Providence, R. I.).

Pólya, G. Heuristic reasoning and the theory of probability. Amer. Math. Monthly **48**, 450–465 (1941). [MF 5538]

In a given mathematical system let the postulates be such that all theorems T, U, V, etc., statable in the system are either provable or disprovable. A mathematician may contemplate T without having been skilful enough to have secured its proof or disproof, and nevertheless hold that T has a definite "subjective probability" or "plausibility" $p(T)$. The author holds that plausibilities $p(T)$, $p(U)$, $p(V)$, etc., obey mathematical laws similar to those of "objective probability" (or frequency) laws which are impersonal, and whose application furnishes the precise rendering of the heuristic common sense of the mathematician engaged in research. Thus if U implies T in the system, $p(T)$ can only be diminished by the subsequent discovery of a disproof of U; and if T implies V, then a subsequent proof of V can only increase $p(T)$ (heuristic induction of T from the discovery of proofs of many of its consequences). Formulas are obtained for the change of $p(T)$ under these circumstances, and are seen to accord with common sense. [While the reviewer has no difficulty with the epistemological side of this paper, he is not clear as to the precise mathematical nature of $p(T)$, $p(U)$, $p(V)$, etc., and their formal combinations. For on the one hand the author asserts that they are not determinate numbers, indeed that the impossibility of assigning fixed numerical values to plausibilities must be held as a fundamental principle. But on the other hand they are elements of a completely ordered set between 0 and 1, may be added, subtracted, multiplied and divided, and obey the laws of total,

compound and relative probability. Now we might have $p(T)=p(\bar{T})$, $p(T)+p(\bar{T})=1$, but seem to be forbidden to set $p(T)=\frac{1}{2}$.] *B. O. Koopman* (New York, N. Y.).

Pearson, Karl. The laws of chance, in relation to thought and conduct. Biometrika **32**, 89–100 (1941). [MF 5740]

The subtitle of the paper reads: "Introductory, definitions and fundamental conceptions, being the first of a series of lectures delivered by Karl Pearson at Gresham College in 1892."

Levi, B. Approximation as an instrument of calculation and proof. Math. Notae **1**, 37–63 (1941). (Spanish) [MF 5606]

Weyl, Hermann. The mathematical way of thinking. Science **92**, 437–446 (1940). [MF 5969]

Denbow, Carl. Means and ends in mathematics. Amer. Math. Monthly **49**, 105–106 (1942). [MF 6243]

Whittaker, E. T. Some disputed questions in the philosophy of the physical sciences. Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 160–175 (1942). [MF 6229] Mainly discussion of Eddington's epistemological principles.

Weinberg, Carlton B. Rigidity, force and physical geometry. Philos. Sci. **8**, 506–532, 618–623 (1941). [MF 5686]

Newsom, C. V. Mathematics and the sciences. Science **94**, 27–31 (1941). [MF 5972]

Michal, Aristotle D. Recent general trends in mathematics. Science **92**, 563–566 (1940). [MF 5970]

Birkhoff, George David. The principle of sufficient reason. Rice Inst. Pamphlet **28**, 24–50 (1941). [MF 6078]

Birkhoff, George David. A mathematical approach to ethics. Rice Inst. Pamphlet **28**, 1–23 (1941). [MF 6077]

TOPOLOGY

***Lefschetz, Solomon. Abstract complexes.** Lectures in Topology, pp. 1–28. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

The algebraic part of the homology theory of complexes has led to the abstraction from geometric complexes of algebraic systems called "abstract complexes" which have a homology theory and, in varying degrees, other properties of their geometric original. After mention of abstract complexes due to Dehn and Heegaard, Newman, Alexander and Mayer, this paper concentrates on the type due to Tucker: a partially ordered set ("on the boundary of" being associated with $<$) in which dimension and incidence number are defined. Both absolute and relative homology and cohomology groups are set up for general coefficient groups leading to the statement of the theorems of duality in their abstract form. All the usual objects of combinatory topology such as subdivision, manifold, etc., are also defined for the abstract complex.

After a brief discussion of chain mappings and the fixed points and coincidences of simplicial transformations, the paper concludes with a discussion of nets of complexes, that is, the abstract part of the Čech homology theory and the theory of infinite complexes. This culminates in the general duality theorem: "the cycles and cocycles of nets of complexes are dual categories." Early in the paper simplicial complexes and polyhedra are described as examples of abstract complexes; at the end, the Alexandroff nerve of a topological space, the barycentric nerve of a geometric complex and the infinite complex illustrate nets of complexes. There is a short bibliography of the principal papers bearing on the topics considered. *W. W. Flexner.*

***Wilder, R. L. Uniform local connectedness.** Lectures in Topology, pp. 29–41. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

A discussion is first given of uniform local connectedness in the ordinary sense (0-ulc), including the classical results, historical remarks and several instructive examples. The extension of this to higher dimensional uniform local connectedness (n-ulc) in the homology sense is outlined and its usefulness in connection with the study of relations between a closed set on its complement in higher dimensional Euclidean spaces and of generalized closed manifolds is indicated. Next uniform local connectedness in the homotopy sense (n-ULC) is discussed along with its applications to similar types of problems in Euclidean spaces. Certain questions concerning the intrinsic structure of generalized closed manifolds are discussed along with their characterizations in terms of uniform locally connectedness properties of their complements. Finally, the related extensions to higher dimensions of the classical property S which have been made by the author are introduced and their usefulness briefly indicated. *G. T. Whyburn.*

***Steenrod, N. E. Regular cycles of compact metric spaces.** Lectures in Topology, pp. 43–55. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

The "dyadic solenoid" is first discussed, and some of its homology groups are computed, as is the one-dimensional homology group, with infinite cycles and integer coefficients, of the complementary space in the 3-sphere. Next, a new type of cycles, the "regular cycles," are introduced; they give more topological properties of the (compact metric) space than do the Vietoris cycles (for example, in the above solenoid); and they obey the Alexander duality theorem. Full details may be found in another paper by Steenrod [Ann. of Math. (2) **41**, 833–851 (1940); cf. these Rev. **2**, 73]. *H. Whitney* (Cambridge, Mass.).

***Eilenberg, Samuel. Extension and classification of continuous mappings.** Lectures in Topology, pp. 57–99. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

Let X and Y be spaces. The totality of continuous mappings $X \to Y$ falls into homotopy classes (two mappings being in the same class if they are deformable one into the other). An outstanding topological problem is the determination of these classes in terms of the invariants of X and Y. The present paper contains a systematic and unified account of the results which bear on this problem and which

center around the work of Hopf, Hurewicz, Whitney and the author. Although details of proof are omitted, the essential steps are clearly indicated and all definitions are stated in extenso.

A natural way to distinguish between homotopy classes of the mappings $X \to Y$ is to examine the homomorphisms which they induce on the homology groups, say, of dimension n. If these homomorphisms correspond to the classes in a one-one manner, (X, Y, n) is called a Hopf system. In recent years cohomology groups have also entered the picture and it is known that under certain conditions the homotopy classes of mappings $X \to Y$ are in one-one correspondence with the n-dimensional cohomology group of X with the Abelian group $\pi_n(Y)$ as coefficient group. When this is the case, (X, Y, n) is called a Whitney system. The definition of the Hopf and Whitney systems is a useful unifying device in the present exposition. Thus a number of the best known theorems can be regarded as descriptions of conditions under which (X, Y, n) is a Hopf or a Whitney system. Typical sufficient conditions include the vanishing of certain homotopy groups of Y and restrictions on the dimension of X. Conditions of the second sort can be replaced by weaker ones, namely, the vanishing of certain cohomology groups of X; this is one of the author's achievements.

Closely related to the class problem is the extension problem: if X' and X'' are subsets of X with $X' \subset X''$, when can a mapping $fX' \subset X''$ be extended to mapping of X''? A sequence of extension theorems, from which a number of the known theorems about homotopy classes can be derived, give sufficient conditions in terms of a characteristic cocycle $c(f)$ in X. Here again the homotopy groups of Y appear to be the most useful coefficient groups. [For a more complete review of the author's extension theorems, see Ann. of Math. (2) **41**, 231–251 (1940); cf. these Rev. **1**, 222. For a review of the author's paper [Ann. of Math. (2) **41**, 662–673 (1940)] on mappings of manifolds into spheres (of which the present work contains a concise account), see these Rev. **2**, 71.] *P. A. Smith* (New York, N. Y.).

***Whitney, Hassler. On the topology of differentiable manifolds.** Lectures in Topology, pp. 101–141. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

In this paper, some of the general theory of sphere bundles (formerly called "sphere spaces") is discussed and illustrated by certain problems pertaining to sets of vector fields in a differentiable manifold. A sphere bundle $\mathfrak{B}$ may be described as follows. (1) There is an underlying topological space T, called the base space. (2) The space T is covered with neighborhoods U_i. (3) Corresponding to each point p on T, there is a set of points $S(p)$, and with each neighborhood U_i containing p there is associated a one-to-one correspondence between $S(p)$ and the points on a fixed r-sphere S_0^r in a Euclidean space E^{r+1}. (4) The sets $S(p)$ and $S(p')$ have no common point if $p \neq p'$. (5) If p is common to two neighborhoods U_i and U_j, the associated correspondences between $S(p)$ and S_0^r induce a self-mapping of S_0^r, which is required to be orthogonal and to vary continuously with p on the intersection of U_i and U_j.

When the base space is a complex, or can be triangulated into one, there are certain characteristic classes which are invariants of $\mathfrak{B}$. These are cohomology classes of various dimensions, the coefficients in the chains being generally either integers or integers mod m. Whitney investigates the normal and tangent sphere bundles of a differentiable n-manifold M^n in an E^m. For example, if $n=2$ and $m=4$, then for each point p on M^2 the set $S(p)$ can be taken as the unit circle in the normal (or the tangent) plane to M^2 at p. There are a number of general results, including a duality theorem for product bundles and also certain formal power series expansions involving the characteristic classes. The paper contains geometric illustrations showing how the general theory can be applied to specific problems. The first of these is the problem of defining, everywhere on a differentiable M^2 in E^4, a continuously varying unit vector normal to M^2. This is always possible when M^2 is not closed or when M^2 is closed and orientable, and it is sometimes possible, sometimes impossible, in other cases. Whitney also determines the characteristic classes for a certain M^2 and a certain M^4, and he deduces that the M^4 cannot be imbedded in E^7. This paper and an earlier one [Bull. Amer. Math. Soc. **43**, 785–805 (1937)] are based on material for which Whitney is preparing a detailed treatment in book form. *S. S. Cairns* (Flushing, N. Y.).

***Cairns, Stewart S. Triangulated manifolds and differentiable manifolds.** Lectures in Topology, pp. 143–157. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

This is an exposition of the more important results (mostly known) on the problems of triangulating manifolds, and the relations between triangulations with various properties which are important in applications. The results are mainly due to Cairns [see Ann. of Math. (2) **41**, 792–808 (1940), and papers referred to there; cf. these Rev. **2**, 71]. *H. Whitney* (Cambridge, Mass.).

***Smith, P. A. Periodic and nearly periodic transformations.** Lectures in Topology, pp. 159–190. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

Let T be a homeomorphism of a connected topological space M with itself. In several papers [Ann. of Math. (2) **35**, 572–578 (1934); ibid. **39**, 127–164 (1938); ibid. **40**, 690–711 (1939); ibid. **42**, 446–458 (1941); these Rev. **1**, 30; **2**, 324; Bull. Amer. Math. Soc. **44**, 497–514 (1938); Amer. J. Math. **63**, 1–8 (1941); these Rev. **2**, 179; with M. Richardson, Ann. of Math. (2) **39**, 611–633 (1938)] the author has studied extensively the properties of periodic T by means of the effects of T on the algebraic structures carried by M. If M is a complex and T is simplicial, the structures are the chains and incidence matrices. For more general M, the algebra is introduced by the method of Čech. If T has period p, studying the effect of T on the basis for the Betti groups of suitable M permits the author here to extend his earlier results on the upper bound of p for given M. He then describes his method of finding homological type invariants for periodic T by means of chain operators determined by T, and gives, using them, (1) a new extension of a theorem of Borsuk [Fund. Math. **20**, 177–190 (1933)] extended by Eilenberg [Duke Math. J. **6**, 428–437 (1940); these Rev. **1**, 319]; and (2) a resume of earlier work on fixed point sets and orbit spaces. T is called nearly periodic (n.p.) if M has a complete system of coverings in the sense of Čech which is invariant under T. It is shown how either a proof that every n.p. transformation operating on a locally Euclidean space is periodic, or a proof that every coset space G/P is n-dimensional, where P is a 0-dimensional compact subgroup of G, would prove that every n-dimensional connected locally Euclidean topological group G

contains one-parameter subgroups. The bearing here of Newman's theorem [Quart. J. Math., Oxford Ser. **2**, 1–8 (1931)] is described. *W. W. Flexner* (Ithaca, N. Y.).

***Zippin, Leo. Transformation groups.** Lectures in Topology, pp. 191–221. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

This paper is largely an account of joint results obtained by the author and the reviewer [cf. these Rev. **2**, 6, 70] with additions and refinements by the author. The topological group G is said to be a topological transformation group of a space E provided to each g in G and x in E there is associated an element $g(x)$ in such a way that the following conditions are satisfied: (1) if e is the identity of G, then $e(x)=x$ for all x in E; (2) $g(g'(x))=(gg')(x)$; (3) $g(x)$ is simultaneously continuous in g and x. The paper begins with examples and general discussion and gives a summary of results on topological groups which are of use later. It is shown that if G is compact and E is any space then points on orbits of dimension at least k form an open set. If a compact G acts effectively on a connected locally Euclidean E and if all orbits are locally connected then G is a Lie group. The proofs of these and all results are merely sketched and not given in full detail. Possibly the principal result is the one which states that there are only two compact connected groups which can operate effectively in Euclidean 3-space and that if coordinates are properly chosen these groups are respectively the group of rotations around an axis and the group of rotations around a point. A necessary and sufficient condition that a one-parameter group in 3-space be a translation group is also found. Some of the preceding results are applied to the foundations of geometry. Hilbert's axioms [appendix 4 in Grundlagen der Geometrie, seventh edition, Leipzig, 1930] giving plane geometries on the basis of a group are shown to be susceptible of weakening and his axioms are carried over as far as possible to 3-space [see also Kerékjártó, Ann. of Math. (2) **29**, 169–179 (1928)]. Following a suggestion of Hilbert [loc. cit.] a concept of rigidity for a general (not topological) transformation group is developed. It is shown that if G has this rigidity property and if it acts on a rather lightly restricted space then G may be topologized and completed so as to become a locally compact topological transformation group of the space. This result is a basic tool in discussing the geometries mentioned above. The result on geometries given by Hilbert and extended here may be roughly stated as follows: Euclidean and hyperbolic geometries are the only geometries in the plane and 3-space which arise from a reasonable group of congruences. *D. Montgomery* (Princeton, N. J.).

***Mac Lane, Saunders and Adkisson, V. W. Extensions of homeomorphisms on the sphere.** Lectures in Topology, pp. 223–236. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

An exposition (concerned mainly with theorems of the authors) of results related to the following problem: If M, M' are Peano spaces on spheres S, S' and $T(M)=M'$ is a topological transformation, under what conditions can T be extended to a homeomorphism from S to S'? The earlier results of Antoine, Gehman, Moore and Schoenflies are cited as well as recent papers of Choquet. The theorems of Adkisson and Mac Lane are differentiated from other results by the use of oriented triods (a notion due to Kline) and by a more intensive study of internal properties of M. Call the points p, q a split pair if $M-p-q$ is the union of two separated sets neither of which is an (open) arc. Further, say that M is triply connected if it has no cutpoints and no split pairs. A very elegant result of the authors is that extension is always possible if M is triply connected [Duke Math. J. **6**, 216–228 (1940); cf. these Rev. **1**, 221]. The problem of extension has combinatorial and group theoretic aspects of considerable interest. At several places in the paper interesting conjectures are mentioned.

C. Chevalley (Princeton, N. J.).

***Harrold, O. G., Jr. The rôle of local separating points in certain problems of continuum structure.** Lectures in Topology, pp. 237–253. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

The author points out the significance of the use of local separating points of continua in certain structural problems, and the unifying effect obtained by this method. The point p of the continuum M is called a local separating point provided there exists a neighborhood R of p such that $M\cdot\bar{R}-p$ is separated between some pair of points of the component of $M\cdot\bar{R}$ containing p. The conditions that the set of local separating points L, and the set of non-local separating points N, be countable give widely different results as indicated by the following: (1) If the set N is countable every connected subset of M is locally connected and is both a G_δ and an F_σ. (2) In order that M contain a totally imperfect connected subset it is necessary and sufficient that it contain a subcontinuum containing but a countable subset of L. (3) In order that a Peano continuum be a hereditary arc sum it is necessary and sufficient that the set N be countable. If the set N is uncountable and each subcontinuum of M contains an uncountable subset of L then M is of finite degree. The author also shows that properties of the set L are related to characteristic types of real-valued continuous functions defined on M. Thirty references are given at the end of the paper. *V. W. Adkisson.*

***Chittenden, E. W. On the reduction of topological functions.** Lectures in Topology, pp. 267–285. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

Introducing in an arbitrary complemented lattice an operation d with some of the formal properties of the derivation of sets in a topological space, the author introduces various definitions and derives some elementary identities about the operations generated by d and the lattice operations. *A. Weil* (Haverford, Pa.).

***Begle, Edward G. Homology local connectedness.** Lectures in Topology, pp. 287–289. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

Homology local connectedness in the dimensions not greater than n (denoted by lc^n) is defined for compacta S in four different ways. These definitions differ in using (a) Vietoris cycles, (b) true cycles, (c) projection cycles, (d) the complex whose n-dimensional simplexes are subsets of S of power $n+1$ and diameter less than α. The principal result is that these definitions are equivalent. The definition (d), due to Alexandroff, is the most direct and apparently the most usable. The author extends to it the Lefschetz method of partial realization [Ann. of Math. (2) **35**, 118–129 (1934)] and states the following theorem: If S is lc^n its homology groups for the dimensions not greater than n are isomorphic with subgroups of the homology groups of a finite complex. Proofs of these statements are in the author's thesis, which will appear shortly in the Annals of Mathematics.

R. H. Fox (Urbana, Ill.).

***Chevalley, Claude. Two theorems on solvable topological groups.** Lectures in Topology, pp. 291–292. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

These important results are stated: If a topological group G is locally compact, connected, locally connected and finite dimensional and if it is solvable, then it is a Lie group. Moreover, if G is not simply connected it contains a toral subgroup T and a subset E homeomorphic to some Cartesian space such that the space of G is the ordinary topological product of the spaces of T and E. In particular, when G is a nilpotent group, the group T is uniquely determined and belongs to the center of G. The first of these theorems extends the work of Pontrjagin on Abelian topological groups, the second extends a theorem of Cartan on solvable, simply connected Lie groups. *L. Zippin.*

***Fox, Ralph H. Topological invariants of the Lusternik-Schnirelmann type.** Lectures in Topology, pp. 293–295. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

A resumé of some results in the theory of category is given. Details may be found in the author's paper, Ann. of Math. (2) **42**, 333–370 (1941); cf. these Rev. **2**, 320. *H. Whitney.*

***Hamilton, O. H. Concerning the decomposition of continua.** Lectures in Topology, pp. 297–298. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

A subcontinuum K of a compact continuum M is said to be a δ-subcontinuum of M if each maximum connected domain of M which contains a point of K is a subset of K, and, furthermore, if K contains each maximum connected domain of M whose boundary it contains. There is apparently some typographical error in this definition as stated in the paper. M is defined to be δ-decomposable if it is the sum of two of its proper δ-subcontinua. The following results are stated by the author. If M is δ-indecomposable and contains no maximum connected domain D such that the boundary of D is the boundary of M then, (1) M is not the sum of a countable number of its δ-composants, (2) no two of the δ-composants of M have a point in common, (3) M is the sum of an uncountable collection of its mutually exclusive δ-composants, (4) the boundary of each composant is a composant of the boundary of M. Other results are somewhat weakened by the restriction that M is not the sum of a finite number $g_1, g_2, \cdots, g_k$ of its δ-subcontinua, such that for each i each boundary point of g_i is a limit point of $M-g_i$. *V. W. Adkisson* (Fayetteville, Ark.).

***Kaplan, Wilfred. Differentiability of regular curve families on the sphere.** Lectures in Topology, pp. 299–301. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

The author states that every family of curves filling the surface of the sphere except for one or two singular points is homeomorphic with a differentiable family. The proof is based on his classification of families of curves [Duke Math. J. **8**, 11–46 (1941); cf. these Rev. **2**, 322]. *H. Whitney* (Cambridge, Mass.).

***Rothe, Erich. On topology in function spaces.** Lectures in Topology, pp. 303–306. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

In recent years problems in analysis have been solved on the basis of topological considerations applied to function spaces. Thus, the existence and uniqueness of solutions for certain functional equations have been deduced from theorems built around the notions of the Kronecker index and the Brouwer degree. The author sketches, without proofs, the development of these theorems and their applications. References to the literature are given. *I. Halperin* (Kingston, Ont.).

***Tukey, John W. Compactness in general spaces.** Lectures in Topology, pp. 307–309. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

In a space X topologized by a closure operator $\bar{H}$ such that $\bar{H} \subset \bar{K}$ if $H \subset K$, the equivalence of the following is stated: (1) If every finite number of a collection of sets meet, then the closures of all the sets meet. (2) If the interiors of a collection of sets cover X, then a finite number of the sets themselves cover X. (3) Every phalanx with values in X has a cluster point in X. (4) Every ultraphalanx with values in X has a limit in X. By a phalanx is meant a function $x(\alpha)$ defined on the class of finite subsets of a set A. The properties of phalanxes are defined in terms of the concept of "decision": $x(\alpha)$ decides for a set H if, for some α_0 and all $\alpha \supset \alpha_0$, $x(\alpha) \varepsilon H$. The phalanx $x(\alpha)$ has x as a limit if $x(\alpha)$ decides for every neighborhood of x; it has x as a cluster point if it decides for the complement of no neighborhood of x. A phalanx is an ultraphalanx if, given any set H, it decides either for H or for the complement of H. Spaces having the above four properties are called compact. That finite sums, closed subspaces, continuous images and product spaces of compact spaces are compact spaces follow from (4). *L. W. Cohen* (Lexington, Ky.).

***van Kampen, E. R. Remark on the address of S. S. Cairns.** Lectures in Topology, pp. 311–313. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

A simplicial subdivision of a k-simplex ($k>2$) is given which has a certain interesting property, discussed in the paper of Cairns [Ann. of Math. (2) **41**, 792–808 (1941); cf. these Rev. **2**, 71]. *H. Whitney* (Cambridge, Mass.).

***Hopf, Heinz. Relations between the fundamental group and the second Betti group.** Lectures in Topology, pp. 315–316. University of Michigan Press, Ann Arbor, Mich., 1941. $3.00.

If $\cap$ is the cap product [Whitney, Ann. of Math. (2) **39**, 397–432 (1938)], the 2-cycles z of a polyhedron P such that $y \cap z \sim 0$ for every one-cocycle with any coefficient domain determine a subgroup V of the second homology group B of P. Let S be the subgroup of B determined by the homology classes of B which contain a continuous image of the 2-sphere. It is then stated without proof that B/V and B/S are entirely dependent on the fundamental group G of P and the nature of these dependences is described. Thus when S is the identity, B depends on G. *W. W. Flexner* (Ithaca, N. Y.).

Menger, Karl. Topology without points. Rice Inst. Pamphlet **27**, 80–107 (1940). [MF 6082]

The purpose of this lecture is to relate results obtained by various authors in attempting to eliminate points as basic concepts in topology. The author first discusses briefly the fundamental concepts of limit, neighborhood and closure as portrayed in the limit class of Fréchet, the neighborhood class of Hausdorff and the closure class of Kuratowski. In each case a set of elements or points con-

stitutes the space. The theories of A. Wald, R. L. Moore and the author, which introduce points as nested sequences of lumps in the manner of introducing real numbers as sequences of rational intervals, are discussed in sufficient detail to give the reader a grasp of the essential details. Other theories introduce points as classes of certain entities with given relations. There is a brief discussion of the theories of M. H. Stone and H. Wallman. The former starts with a Boolean ring, the latter with a distributive lattice. A slightly more detailed study of a theory of A. N. Milgram, based on a partially ordered set satisfying certain simple conditions, shows the relation of these results to those of Stone and Wallman. The author points out that the methods of Stone and Wallman are applicable to spaces in which the original sequence definition of point would not work, while Milgram's method gives results which are not point spaces at all. *V. W. Adkisson* (Fayetteville, Ark.).

Lubben, R. G. Concerning the decomposition and amalgamation of points, upper semi-continuous collections, and topological extensions. Trans. Amer. Math. Soc. 49, 410–466 (1941). [MF 4340]

The author makes an extended and detailed study of topics in that part of abstract topology sometimes called the "geometry of lumps." On the one hand points are not regarded as necessarily indivisible; on the other, ideal points may be created as infinite collections of subsets of the basic space. In Chapter I order relations among these ideal points are discussed. A theory of accessibility, formally analogous to Carathéodory's, is introduced. Chapter II is devoted to topological relations among ideal points. Spaces in which the complement of each point is normal are called "semi-completely normal" and several precise results are obtained for such spaces. Decompositions of spaces, in particular, upper semi-continuous decompositions, and extensions of spaces to compact imbedding spaces are then treated. In Chapter III extensive use is made of lattice theory to get further results, especially for spaces which are not semi-completely normal. *H. Wallman.*

Ribeiro, Hugo. Une extension de la notion de convergence. Portugaliae Math. 2, 153–161 (1941). [MF 4879]

Let A be an ordered set. Then A is said to be oriented if $a \varepsilon A$ implies an $a' \varepsilon A$ such that $a' > a$, and $B \subset A$ is said to be cofinal with A if $a \varepsilon A$ implies a $b \varepsilon B$ such that $b > a$. Considering a function on A to a space 1, the author uses the notation $x(a|A)$ for such a function, $x(a)$ and $x(A)$ a value and the range of this function, respectively. If 1 is a space (V) such that each superset of a neighborhood is a neighborhood, a function $x(a|A)$ is said to converge to $x \varepsilon 1$, in notation $x(a|A) \to x$, if for each neighborhood I_x^0 of x there exists an $a_0 \varepsilon A$ such that $a > a_0$ implies $x(a) \varepsilon I_x^0$. The author then proves: (Theorem) A necessary and sufficient condition that $U \subset 1$ be a neighborhood of a point $x \varepsilon 1$ is that each function $x(a|A)$ such that $U \cap x(A) = 0$ does not converge to x. Also, (α) if $x(a) = x$ for each $a \varepsilon A$, then $x(a|A) \to x$, and (ϵ) $x(a|A) \to x$ if and only if, corresponding to each B cofinal with A, there exists an oriented system N and a function $b(n|N)$ on N to B such that $x(b(n)|b(N)) \to x$.

Now with 1 an arbitrary set, let it be decided which functions on A (oriented) to 1 converge. If this convergence is such that (α) and (ϵ) hold the author gives the definition: $U \subset 1$ is a neighborhood of a point x if there is no function $x(a|A)$ which converges to x and $U \cap x(A) = 0$. By this definition it is shown that a space (V) is obtained in which a superset of a neighborhood is a neighborhood. The author also discusses the relation of convergence to closure, continuity and extensions. *J. F. Randolph* (Ithaca, N. Y.).

Tola P., José. On the equivalence of the two forms of continuity of operations, for sequences and for neighborhoods, in topological spaces. Thesis, Universidad Nacional Mayor de San Marcos, Lima, 1941. 31 pp. (Spanish) [MF 5927]

The present paper contains a detailed exposition, with proofs, of theorems announced previously [Actas Acad. Ci. Lima 4, 73–75 (1941); these Rev. 3, 56]. Additional theorems of the same nature and pertinent examples are also included. *J. V. Wehausen* (Columbia, Mo.).

Moore, R. L. Concerning separability. Proc. Nat. Acad. Sci. U.S.A. 28, 56–58 (1942). [MF 6137]

It is shown that any space satisfying axioms 0 and 1 of the author's "Foundations of Point Set Theory" [Amer. Math. Soc. Colloquium Publ., v. 13, New York, 1932] is separable provided it contains no uncountable collection of disjoint domains. Also, examples are given showing the existence of both separable and non-separable non-metric spaces satisfying the author's axioms $0-4$, 5_1, 5_2 and 6. *G. T. Whyburn* (Charlottesville, Va.).

Kozloff, Zoé. Sur la séparabilité multiple. C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 110–114 (1940). [MF 3213]

Given a space X, a class (U) of subsets of X, the class (CU) of the complements of sets U and $M(U) = (U)(CU)$, multiple separability in X is described by any axiom in one of the following four groups. I_k: If $(E_n) \subset (U)$ is a set of k sets ($k = 2, 3, \cdots, \infty$) such that $\Pi_{n=1}^{k} E_n = 0$, then there exist k sets $H_n \varepsilon M(U)$ such that $H_n \supset E_n$ and $\Pi_{n=1}^{k} H_n = 0$. $\mathrm{I}_{\overline{\lim}}$, $\mathrm{I}_{\underline{\lim}}$ are obtained from I_k by replacing the operation $\Pi_{n=1}^{k}$ with $\overline{\lim}$, $\underline{\lim}$. II_k: If $(E_n) \subset (U)$ is a set of k sets, then there exist k sets $H_n \varepsilon (CU)$ such that $H_n \supset E_n - \Pi_{m=1}^{k} E_m$ and $\Pi_{n=1}^{k} H_n = 0$. $\mathrm{II}_{\overline{\lim}}$, $\mathrm{II}_{\underline{\lim}}$ are obtained from II_k by replacing $\Pi_{n=1}^{k}$ with $\overline{\lim}$, $\underline{\lim}$. III_k, $\mathrm{III}_{\overline{\lim}}$, $\mathrm{III}_{\underline{\lim}}$ are obtained from group II by replacing the condition $H_n \varepsilon (CU)$ with $H_n \varepsilon (U)$. IV_k, $\mathrm{IV}_{\overline{\lim}}$, $\mathrm{IV}_{\underline{\lim}}$ are obtained from group II by requiring that the sets in (U) be mutually exclusive. Eleven theorems of the following type are stated: If (U) is a σ-system, then IV_k, k finite, implies IV_∞, $\mathrm{IV}_{\overline{\lim}}$, $\mathrm{IV}_{\underline{\lim}}$. If (U) is a σ-system which contains the intersection of any finite number of U, then III_∞ implies $\mathrm{III}_{\overline{\lim}}$. A table is appended showing, except in twelve cases, which properties of multiple separability for $k = 2$, finite, ∞, $\overline{\lim}$, $\underline{\lim}$ are enjoyed by the classes of sets A, their complements, additive and multiplicative Baire classes; for example, (A) has no property in group III. These results extend the work of P. Novikoff [C. R. (Doklady) Acad. Sci. URSS 2, 273–276; 3, 145–148, 557–560; 4, 8–11 (1934)], S. Ruziewitch [Fund. Math. 24, 199–205 (1935)] and W. Sierpinski [Fund. Math. 23, 292–303 (1934)]. *L. W. Cohen* (Lexington, Ky.).

Bourbaki, Nicolas. Espaces minimaux et espaces complètement séparés. C. R. Acad. Sci. Paris 212, 215–218 (1941). [MF 4896]

It is known that every compact topological space E has the following property: every one-to-one and continuous map of E on a Hausdorff space (called a separated topological space in this paper) is bicontinuous [see the author's

Éléments de Mathématique, Livre III, Chap. I, p. 63 in Actual. Sci. Ind., no. 858, Paris, 1940; cf. these Rev. 3, 55, and Alexandroff-Hopf, Topologie I, p. 95, Berlin, 1935]. In this paper the author studies all separated topological spaces E with the same property. An equivalent property is the following: every separated topology which is less fine than the given topology of E is equivalent to it. For this reason the spaces with the property in question are called minimal separated spaces. The following theorem is established: A necessary and sufficient condition that a separated space E be minimal is that every filter which has open sets of E as a base and which has at most one adherent point be convergent. [For definitions and properties of filters see the author's Éléments de Mathématique and H. Cartan, C. R. Acad. Sci. Paris **205**, 595–598, 777–779 (1937).] The condition stated consists of two parts: (1) Every filter having a base of open sets has at least one adherent point. (2) Every filter having a base of open sets and possessing a single adherent point converges to that point. The condition (1) is necessary and sufficient that E be absolutely closed [H-closed; see Alexandroff-Hopf, p. 90]; examples show that it is not sufficient that E be minimal. The author next characterizes the minimal separated spaces E which are not compact. In such a space there exists an ultrafilter which is not convergent. A space is called completely separated if there exists no pair of distinct points a and b such that every closed neighborhood of a meets every closed neighborhood of b. The results established show that every completely separated minimal topological space is compact. An example is given to show that there does not necessarily exist a compact topology less fine than the topology of a separated space E even if E is metrizable. Finally, a sufficient condition that a space be completely separated is that there exist, for each pair of distinct points a and b, a continuous numerically-valued function which takes on distinct values at a and b. A necessary and sufficient condition that a space E have the latter property is that the topology of E be finer than a completely regular topology of the space. *G. B. Price.*

Fan, Ky. Espaces quasi réguliers, quasi normaux et quasi distanciés. C. R. Acad. Sci. Paris **211**, 348–351 (1940). [MF 5352]

Chittenden [Trans. Amer. Math. Soc. **31**, 314 (1929)] has generalized a theorem about normal spaces first proved by Urysohn to more general spaces. The author extends other known theorems on regular and normal spaces to spaces which he calls quasi-regular and quasi-normal. There are also theorems on symmetric spaces. A symmetric space is a space in which, if any given point is in the closure of a second point, then the second point is in the closure of the first. *D. Montgomery* (Princeton, N. J.).

Fomin, S. V. Extensions of topological spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 114–116 (1941). [MF 5854]

An extension of a T_3-space R is a T_2-space with an everywhere dense subset homeomorphic to R. Every such extension can be defined by putting its points into a one-to-one correspondence with certain families of open subsets of R; taking the latter subsets out of a given basis of R, the author defines and studies various rather general types of extensions. Use is made of a new type of continuity, which is the same as the usual one for a mapping into a regular space, but is more general otherwise. *A. Weil.*

Bourgin, D. G. Some properties of real linear topological spaces. Proc. Nat. Acad. Sci. U.S.A. **27**, 539–544 (1941). [MF 5735]

Let L denote a real linear topological space (l. t. s.) in the sense of Hyers' modification [Revista Ci., Lima **41**, 555–574 (1939); cf. these Rev. **1**, 318] of the von Neumann axioms [Trans. Amer. Math. Soc. **37**, 1–20 (1935)], and let L^* denote the set of all linear functionals on L. This paper states 8 lemmas and 24 theorems for spaces L and L^*; a complete discussion of the results and their detailed proofs will be published elsewhere. The first section is devoted to preliminary results; these concern the nature of neighborhoods, closure, convexity, the existence of pseudo norms with certain properties, and regularly L convex sets. The second section considers the weak topology of L. The weak neighborhoods in L and L^* are defined in the usual way. The weak topology of L as elements and of L^* as functionals is indicated by the symbols $w(L)$ and $w^*(L^*)$, respectively. In general it is not true that L is a l.t.s. in its $w(L)$ topology. Let $\Theta=\{x \mid f(x)=0 \text{ for all } f\in L^*\}$. Let $X=x+\Theta$. The sets Θ, X are considered as elements of a new space J. Neighborhoods are defined for J; the space J is a completely regular, Hausdorff, locally convex l.t.s. There is an analysis of the set Θ. If A is $w(L)$ closed, it is closed. The following conditions on K are equivalent: (a) K is regularly L closed; (b) K is convex and $w(L)$ closed; (c) a condition in terms of J. The next section gives results which hold when L is locally convex. The following generalization of a result due to Mazur is a sample: if K is convex and closed, then K is $w(L)$ closed. The fourth section defines the notion of outer countable basis for L and in terms of it gives a sufficient condition for the metrizability of L^*; if L is locally convex, the condition is also necessary. The fifth section assumes that L is locally bounded. Hyers has established the existence of a pseudo-norm satisfying five conditions; two of them are made significantly sharper in the present paper. The sixth section treats the conjugate space L^*. In the seventh section it is assumed that L is a Banach space. $w(L)$ completeness of order $\aleph_a$ is defined by requiring the directed sets in Goldstine's definition of $w(L)$ completeness to have power $\aleph_a$. If the unit sphere is $w(L)$ bicompact of order $\aleph_a$ (in the sense of Alexandroff and Urysohn), it is $w(L)$ complete of order $\aleph_a$. Other results concern weak convergence topologies. The notion of Helly property of order $\aleph_a$ is defined, and two theorems involving it are stated. Finally, three theorems indicate relations between various compactness notions. *G. B. Price.*

Šura-Bura, M. Zur Theorie der bikompakten Räume. Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 385–388 (1941). (Russian. German summary) [MF 4561]

The author extends the following theorems, well-known for compact metric spaces, to (bi-)compact Hausdorff spaces. Comp(E) will denote any component of E and Int(E) its interior. (I) If K is the intersection of all the sets containing x which are both open and closed in a (bi-)compact Hausdorff space R then K is the component of x in R. (II) (Generalized theorem of Janiszewski) If F is a closed subset of a Hausdorff space R, if H is the common boundry of F and $G=R-F$, and if the (bi-)compact set C is connected and meets H, then: (i) $\text{Comp}(C\cap F)$ meets H; (ii) if C meets G, then H meets $\overline{\text{Comp}(C\cap G)}$; (iii) if $\text{Int}(C)$ meets G, then H meets $\overline{\text{Comp}(C\cap \text{Int}(C))}$. (III) (Generalized

theorem of Sierpiński) A connected (bi-)compact Hausdorff space is not the sum of countably many disjoint nonempty closed sets. *J. W. Tukey* (Princeton, N. J.).

Parhomenko, A. Über eineindeutige stetige Abbildungen auf kompakte Räume. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 225–232 (1941). (Russian. German summary) [MF 5173]

Beispiele von Punktmengen, die gleichzeitig (absolute) F_σ- und G_δ-Mengen sind, und auf kein Kompaktum eineindeutig und stetig abgebildet werden können. Satz I. Jeder im Kleinen bikompakte Hausdorffsche Raum R kann auf ein Bikompaktum (und falls R eine abzählbare Basis besitzt, sogar auf ein Kompaktum) eineindeutig und stetig abgebildet werden. Satz II. Ist R ein absolutes F_σ, das eineindeutig und stetig auf ein Kompaktum abgebildet werden kann, so ist die offene Menge aller Punkte, in denen R kompakt ist, überalldicht in R. Satz III. Die Vereinigungsmenge von endlichvielen offenen Simplexen (beliebiger Dimensionszahlen) eines R^n kann auf ein Kompaktum eineindeutig und stetig abgebildet werden.

Author's summary.

Wilder, R. L. Decompositions of compact metric spaces. Amer. J. Math. **63**, 691–697 (1941). [MF 5616]

For a local topological property ψ in a compact metric space M a point of M where ψ fails to hold is called a ψ-singular point; and ψ is said to be expansive relative to a given class Γ of compact metric spaces with $M \in \Gamma$ provided the set S of all ψ-singular points of M is either empty or such that $\bar{S}$ has a nondegenerate component. The components of $\bar{S}$ and the individual points of $M-\bar{S}$ are called the ψ-prime parts of M. It is shown first that the space M' of ψ-prime parts of M has no ψ-singular points if $M' \in \Gamma$ (this being an extension of a theorem of the reviewer). Several results concerned with expansiveness of properties relating to higher dimensional local connectedness in the homology sense are proved, culminating in the theorem that, if r and n are integers such that $-2<r<n$, the property ψ^n_{r+1} of being i-locally connected for $i=r+1, r+2, \cdots, n$ is expansive relative to the class $C_r{}^n$ of compact metric spaces M which are i-locally connected for $i \leqq r$ and have finite Betti numbers $p^i(M)$ for $r<i \leqq n$. Further, if $M \in C_r{}^n$ and the ψ^n_{r+1}-prime parts of M are simply i-connected for $i=1, 2, \cdots, n-1$, the space of ψ^n_{r+1}-prime parts of M is i-locally connected for $i \leqq n$. By way of applications, these results are shown to yield conditions under which the space of ψ-prime parts of M will be, respectively, a 2-dimensional manifold or a generalized closed n-manifold.

G. T. Whyburn (Charlottesville, Va.).

Ochan, J. S. Space of subsets of a topological space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 107–109 (1941). [MF 5852]

If E is a topological space, the set $\mathcal{S}$ of all subsets is topologized as follows: If F is a closed set and G is any open set containing F, the neighborhood $U_G(F)$ is the set of all sets M such that $F \subset M \subset G$. If K is an arbitrary element of $\mathcal{S}$, its neighborhoods consist of all those of the first type of which it is a member. Then, if E is a T_1-space, $\mathcal{S}$ is a zero-dimensional completely regular space. If E is a bicompact space of character τ, $\mathcal{S}$ has no more than τ mutually disjoint neighborhoods. A continuous biunique mapping of E_1 into E_2 induces a mapping of $\mathcal{S}_1$ into $\mathcal{S}_2$ which is also continuous and biunique. The additive function $f(M)=A \cdot M+B$ from $\mathcal{S}$ to $\mathcal{S}$ is continuous if and only if $A-B$ is closed in E. An "elementary T_1-space of cardinal number τ" is defined as a set of power τ in which the neighborhoods of any element are the sets containing it whose complements are finite. For such spaces $\mathcal{S}$ is bicompact. In addition, if E is not such a space, $\mathcal{S}$ is not compact. The author finally considers the space $\mathcal{S}_F$ of closed sets of E topologized as above and the space A_F of closed sets of E topologized according to a method of P. Alexandroff involving minimal coverings. If E is a T_1-space, $\mathcal{S}_F$ may be mapped biuniquely and continuously into A_F. *J. V. Wehausen* (Columbia, Mo.).

Fan, Ky. Sur les types homogènes de dimensions. C. R. Acad. Sci. Paris **211**, 175–177 (1940). [MF 5343]

Fréchet has introduced the concepts of dimensional type and local dimensional type [Les Espaces Abstraits, Paris, 1928]. Let E and F be two metric spaces and let a and b be two points in E and F, respectively. The local dimension of E at a is at most equal to the local dimension of F at b ($d_aE \leqq d_bF$) if for every neighborhood $V(b)$ there is a neighborhood $V(a)$ homeomorphic to a part of $V(b)$. A space E is then called dimensionally homogeneous if $d_aE=d_bE$ for every a and b in E. Given two dimensionally homogeneous spaces E and F we say that the homogeneous type of dimension of E is at most equal to that of F if for a point a in E and a point b in F we have $d_aE \leqq d_bF$. The note is concerned with a number of theorems concerning this concept.

D. Montgomery (Princeton, N. J.).

Roberts, J. H. A theorem on dimension. Duke Math. J. **8**, 565–574 (1941). [MF 5193]

A separable metric n-dimensional space M can be topologically imbedded in Euclidean $(2n+1)$-space R_{2n+1} in such a way that every linear subspace R, whose dimension is greater than n, intersects M in a set of dimension not exceeding dim $R-n-1$. This theorem is applied (1) to give a short proof of a theorem of Hurewicz [S.-B. Preuss. Akad. Wiss. **1933**, nos. 24/25, 754–768]: a separable metric space M can be continuously mapped into any Euclidean space R, whose dimension does not exceed dim M, in such a way that inverse images of points have dimension not exceeding dim $M-$dim R; (2) to prove a conjecture of Hurewicz [Nederl. Akad. Wetensch., Proc. **30**, 161, footnote 7 (1927)]: if M is a separable metric space and $n<$dim M then there is an n-dimensional space which can be mapped continuously onto M in such a way that inverse images of points have power not exceeding dim $M-n+1$.

R. H. Fox (Urbana, Ill.).

Eilenberg, Samuel and Miller, E. W. Zero-dimensional families of sets. Bull. Amer. Math. Soc. **47**, 921–923 (1941). [MF 5938]

Theorem: let X be a unicoherent Peano space and $\{A_\alpha\}$ a 0-dimensional family of subsets (each A_α has arbitrarily small neighborhoods whose boundaries do not meet $\sum A_\alpha$). If every A_α fails to cut X between given points x_1 and x_2, so does $\sum A_\alpha$. Corollaries: several theorems about upper-semi-continuous collections. *P. A. Smith.*

Blumenthal, Leonard M. A new concept in distance geometry with applications to spherical subsets. Bull. Amer. Math. Soc. **47**, 435–443 (1941). [MF 4534]

The author is again concerned with the embedding problem for semi-metric spaces, that is, sets in which a distance pq is defined for which $pq=qp$, and $pq=0$ if and

only if $p=q$. A semi-metric space S has σ-relative congruence indices (n, k) with respect to a given class (Σ) of semimetric spaces if (i) $\Sigma\epsilon(\Sigma)$, (ii) each n points of Σ can be congruently embedded in σ and (iii) Σ has at least $n+k$ points together imply that Σ can be congruently embedded in S; σ is then also called a catalytic space of index (n, k) of S with respect to (Σ). This generalizes the notions of congruence order [Menger, Nederl. Akad. Wetensch., Proc. 30, 1–5 (1927)] and quasi-congruence order [Menger, Amer. J. Math. 53, 721–745 (1931)].

By extending certain recent results [Blumenthal and Thurman, Amer. J. Math. 62, 835–854 (1940); these Rev. 2, 72] about embeddability in $S_{n,r}$, which is the surface of a sphere of radius r in $n+1$ dimensional Euclidean space with distances measured along great circles, the author is able to prove that (i) the open hemisphere of $S_{n,r}$, and (ii) any smaller spherical cap of $S_{n,r}$, has congruence indices $(n+2, n)$ with respect to all semi-metric spaces. If the cap is contained within an open cap of geodesic radius $r\cdot\cos^{-1}[(n+1)^{-1}]$, then (iii) the cap has congruence indices $(n+2, n-1)$. The result (iii) is best possible for $n=2$, but not for $n=3, 4$. *J. W. Tukey* (Princeton, N. J.).

Swingle, Paul M. Indecomposable connexes. Bull. Amer. Math. Soc. 47, 796–803 (1941). [MF 5491]

An indecomposable connexe is a connected set M such that for every split into connected sets $M=H+K$ we have $\bar{H}=\bar{M}$ or $\bar{K}=\bar{M}$. This is a generalization of indecomposable continuum and the paper gives results analogous to most of the well-known theorems on indecomposable continua. Of particular interest is the following generalization of the three point theorem on indecomposable continua: If M is connected and $\bar{M}$ is compact, then M is an indecomposable connexe if and only if there are three distinct points such that M is an irreducible joining connexe closure between any two of them. A connected set K is an irreducible joining connexe closure between x and y if K contains a connected subset L such that $L+x+y$ is connected and $\bar{L}=\bar{K}$ for all such sets L. *W. L. Ayres* (Lafayette, Ind.).

Ollmann, Loyal F. Solution of a problem of Ayres. Amer. J. Math. 64, 61–71 (1942). [MF 5989]

A subset K of a space M is said to be 3-connected in M provided no two points of K can be separated in M by two points of M. In a previous paper [Fund. Math. 19, 79–91 (1932)] W. L. Ayres proposed the following problem: If K is any finite subset of a compact planar locally connected continuum M which is 3-connected in M, does there exist a simple closed curve J in M containing K? In the same paper he proved that the answer to the question is positive provided K consists of three points, and in later unpublished work he showed that the answer is negative for sets K consisting of eleven or more points. The present paper answers the question in the affirmative for any set K containing not more than five points. An example shows that the theorem is false if K contains six or more points even under the stronger hypothesis that the set M itself is three-connected. Additional examples show the necessity of each of the following conditions: (a) M is a subset of the plane; (b) M is three connected; (c) M is a locally connected continuum (the weaker condition that M be arcwise connected is not sufficient). The paper concludes with the result that, for any positive integer m greater than 2, there exists a compact planar locally connected continuum M containing a set of $2m$ points which is three-connected in M and cannot be included in $m-2$ mutually exclusive simple closed curves in M. *D. W. Hall* (Providence, R. I.).

Harrold, O. G., Jr. Continua of finite sections. Duke Math. J. 8, 682–688 (1941). [MF 5952]

This paper characterizes the continua on which a real valued continuous function of finite sections can be defined, that is, a real valued continuous function f such that, for each $y, f^{-1}(y)$ is a finite set. The following theorem is proved. The continuum M has the above property if and only if M is locally connected and every dendrite in M has the property. The author proves only the sufficiency of these conditions since the necessity follows from results established by Čech, and the fact that if M has the property every subcontinuum has this property. It is shown that there exists a map $f(M)=(0, 1)$ such that $f^{-1}(y)$ is finite for all but a countable set of values y and for these values $f^{-1}(y)$ is countable. Then the desired map $g(M)=(0, 1)$ is obtained by modification of f in the neighborhood of $f^{-1}(y)$.

V. W. Adkisson (Fayetteville, Ark.).

Wagner, K. Charakterisierung stetiger Kurven mit Hilfe eines allgemeinen Richtungsbegriffs für Punktmengen. Math. Ann. 117, 672–686 (1941). [MF 5651]

The sets considered in this paper are plane sets, the δ-neighborhood $U(p; \delta)$ of a point p is the circular disc (boundary included) with center p and radius δ, and a sector of $U(p; \delta)$ of opening $\varphi_2-\varphi_1$ consists of all points with polar coordinates (r, φ) satisfying the relations $0\leqq r\leqq\delta$, $\varphi_1<\varphi<\varphi_2$, with p the pole of the system. Two such sectors are mutually exclusive provided they have only p in common, and a family $\mathfrak{S}$ of pairwise mutually exclusive sectors forms a covering of fineness ϵ of a set P in the δ-neighborhood of one of its points p provided the product $U(p; \delta)\cdot P$ is covered by $\mathfrak{S}$ and each sector of $\mathfrak{S}$ has opening at most ϵ. It is evident that $\mathfrak{S}$ is at most denumerable and the author shows that, if $\epsilon>0$ implies $\delta>0$ such that $\mathfrak{S}$ is a covering of P of fineness ϵ in the δ-neighborhood of p, then there is a finite family $\mathfrak{S}$ with this property. This is introductory to the fundamental definition: The set P is differentiable at one of its points p provided corresponding to each positive ϵ there is a positive δ such that a finite family $\mathfrak{S}$ of fineness ϵ exists covering P in the δ-neighborhood of p. This concept is made the basis for characterizing continuous curves, that is, continuous images of a closed segment. The author proves (1) every bounded (plane) continuum which is differentiable at each of its points is a continuous curve and (2) if a neighborhood of a point p of a continuum K exists such that K is differentiable at each of its points which lie in this neighborhood, then there exists a second neighborhood of p and a continuous curve (each point of which is a point of K) which contains all the points of K in this neighborhood. After proving that a continuum K is locally connected at one of its points p provided there exists a neighborhood of p such that K is differentiable at each of its points in this neighborhood, the characterization theorem (1) follows at once from the well-known result of Hahn-Mazurkiewicz. The author prefers, however, to establish his theorem independently and, in effect, gives a simplified proof of the Hahn-Mazurkiewicz theorem which is permitted by his special assumptions. The treatment is quite elementary. The author fails to mention the obvious relations between his investigations

and those of G. Bouligand [cf. Géométrie infinitésimale directe, Vuibert, Paris, 1932].

It should also be observed that the title of the paper is somewhat misleading, for since there exist bounded continuous curves (for example, the Peano-Hilbert curve) which are not differentiable at any point, the property of differentiability at each point is not characteristic for such sets. *L. M. Blumenthal* (Columbia, Mo.).

Kaplan, Wilfred. The structure of a curve-family on a surface in the neighborhood of an isolated singularity. Amer. J. Math. **64**, 1–35 (1942). [MF 5986]

Continuing work in previous papers [Duke Math. J. **7**, 154–185 (1940); **8**, 11–46 (1941); these Rev. **2**, 322], the author classifies topologically the structures of regular families of curves filling a sphere except for two singular points N and S, and (as an immediate consequence) the structures of families on a surface in the neighborhood of any singular point. In the first problem, three cases arise: (a) there is a curve joining N and S; (b) there is none, nor is there a closed curve in the family; (c) there is a closed curve in the family. In (a), cutting along a curve from N to S gives a family filling a strip in the plane; this case was taken care of in a previous paper. In the other cases, the "flowers" at N(and S) are studied; each consists of a set of curves with both ends at N, no one separating another from S. The full classification is too complicated to be given here. *H. Whitney* (Cambridge, Mass.).

Wecken, Franz. Fixpunktklassen. I. Math. Ann. **117**, 659–671 (1941). [MF 5650]

Let f be a continuous mapping of a finite polyhedron $\mathfrak{P}$ onto itself. The fixed points of f fall into fixed-point classes [Nielsen], p_1 and p_2 belonging to the same class if $f(\tau)$ is homotopic to τ for some arc $\tau = p_1p_2$. To each f.p.-class of f there is associated an index (=sum of indices of individual fixed points when they are isolated and $\mathfrak{P}$ is locally Euclidean). Theorem: if mappings f, g belong to the same homotopy class, the f.p.-classes of f with non-zero indices can be put into one-one correspondence with those of g in such a way that corresponding classes have equal indices. *P. A. Smith* (New York, N. Y.).

Scorza Dragoni, Giuseppe. Über die fixpunktfreien Abbildungen der Ebene. Abh. Math. Sem. Hansischen Univ. **14**, 1–21 (1941). [MF 5439]

This review of the known facts about fixed-point-free transformations of S into itself (S a plane or a plane circular ring) includes proofs of Brouwer's "translation theorem" and Poincaré's "last geometric theorem," based on various preliminary theorems of Brouwer, Kerékjártó and the author (stated without proofs). *P. A. Smith.*

Miller, D. D. Extension and reduction theorems for certain types of continuous transformations. Amer. J. Math. **64**, 215–228 (1942). [MF 6000]

This paper deals with a compact locally connected continuum A and various types of single valued continuous transformations $T(A)=B$. The theorems of the paper may be conveniently divided into three groups. In the first group we have results such as the following: In order that a transformation having property P on A have property P on every subcontinuum of A each of the following conditions is both necessary and sufficient: (a) T is A-set reversing, where P is monotone; (b) T is non-alternating on every subcontinuum of A, where P is monotone; (c) T is homeomorphic or constant on each cyclic element of A, where P is monotone or non-alternating; (d) T is homeomorphic or constant on A, where P is interior; (e) B is a single point, where P is non-separating. The author does not use any of the recent results on arc-preserving transformations [see D. W. Hall and G. T. Whyburn, Trans. Amer. Math. Soc. **48**, 63–71 (1940); these Rev. **1**, 319]. These results would have considerably simplified the proofs of some of the above theorems, since any transformation having property P on every subcontinuum of A is, of course, arc-preserving on A, where P is monotone, non-alternating, interior or non-separating.

The second group of theorems characterizes the A-sets in A by means of the transformations which may be defined upon them. It is in this section that the author brings in the generalizations introduced by E. P. Vance [Duke Math. J. **6**, 66–79 (1940); these Rev. **1**, 223] of non-alternating and non-separating transformations as previously introduced by G. T. Whyburn [Amer. J. Math. **56**, 294–302 (1934)] and J. F. Wardwell [Duke Math. J. **2**, 745–75ο (1936)]. He obtains the following results: (1) If R is a subcontinuum of A then the following properties are equivalent: (a) R is an A-set; (b) every transformation having property P on A has property P on R, where P is monotone, non-alternating or non-separating; (c) every transformation completely non-alternating on A is non-alternating on R. (2) If a transformation has property P on A then it has the same property on every A-set in A, where P is completely non-alternating, weakly non-alternating, completely non-separating or weakly non-separating.

The final group of theorems makes use of the concept of a basis set of a locally connected continuum as introduced by W. L. Ayres [Trans. Amer. Math. Soc. **30**, 567–578 (1928)], to obtain conditions under which a transformation $T(A)=B$ having property P on certain subsets of A must have property P on A itself. The result proved states that, if K is a basic set of A and if T has property P on the cyclic chain $C(x, y)$ for every pair of points x and y of K, then T has property P on A, where P is monotone, non-alternating or non-separating. *D. W. Hall* (Providence, R. I.).

Hall, D. W. and Kelley, J. L. Periodic types of transformations. Duke Math. J. **8**, 625–630 (1941). [MF 5947]

If X is a compact metric space and T is pointwise periodic on X, then the following are equivalent: (a) the collection of point orbits is a continuous collection; (b) T is strongly almost periodic; (c) T is regular; (d) the orbit of every closed set is closed. The transformation T is strongly almost periodic if for any ϵ there is an integer K such that every block of K successive integers contains an integer n such that $\rho(x, T^n(x))<\epsilon$ for every x in X; T is regular if for any ϵ there is a δ such that $\rho(x, y)<\delta$ implies $\rho(T^n(x), T^n(y))<\epsilon$ for all n. If T is pointwise periodic on the locally connected continuum X in the sphere S^3 such that no two points of X separate X, then T may be extended to a pointwise periodic transformation of S^3. It follows then that T is periodic and the transformations may be completely classified. Finally it is shown that, if T is any continuous transformation of a compact metric space X into a subset of itself, then X contains a finite set on which T is periodic or X contains a perfect set on which T is pointwise almost periodic.

W. L. Ayres (Lafayette, Ind.).

Martin, Venable. Monotone transformations of non-compact two-dimensional manifolds. Duke Math. J. **8**, 136–153 (1941). [MF 3949]

In extension of the results of Roberts and Steenrod [Ann. of Math. (2) **39**, 851–862 (1938)] concerning compact manifolds the author studies upper semi-continuous collections of compact continua filling a non-compact 2-manifold M. He characterizes spaces generated by such collections (that is, continuous images of 2-manifolds under monotone transformations) and shows in particular that, if one-dimensional Betti numbers mod 2 vanish for all the continua of the collection, the space of decomposition is homeomorphic to M. Furthermore he gives several results referring to collections of non-compact continua. Details of some of the proofs are difficult to follow. *W. Hurewicz.*

Schneckenburger, Edith R. On 1-bounding monotonic transformations which are equivalent to homeomorphisms. Amer. J. Math. **63**, 768–776 (1941). [MF 5624]

It has been shown by Roberts and Steenrod [Ann. of Math. (2) **39**, 851–862 (1938)] that, if A is a compact two-manifold and T is a monotonic transformation such that each inverse set $T^{-1}(a)$ has 1-dimensional Betti number zero, then $T(A)$ is homeomorphic with A. The present paper considers the question of finding the spaces A, in addition to two-manifolds, such that $T(A)$ is homeomorphic to A for every such T. The main result is that, for cyclic Peano spaces imbedded in two-manifolds, the only possible spaces A are the simple closed curve and the entire manifold. An acyclic Peano space has this property if and only if it is an arc. Other special cases are considered. *W. L. Ayres.*

Kagno, I. N. On a certain non-separating graph on an orientable surface. J. Math. Phys. Mass. Inst. Tech. **20**, 370–387 (1941). [MF 6107]

Let G be a connected linear graph without free arcs or chains, on an orientable surface Σ_p of genus $p>0$, such that $\Sigma_p - G$ is a 2-cell. Then, if α_1 is the number of arcs or simple chains in G, $2p \leqq \alpha_1 \leqq 6p-3$. The possible structures of G are studied. An application to uniformization theory is given. *H. Whitney* (Cambridge, Mass.).

Chojnacki, Chaim. A contribution to the four color problem. Amer. J. Math. **64**, 36–54 (1942). [MF 5987]

The four color problem is easily reduced to the consideration of a standard map, in which every polygon has at least five sides and every vertex belongs to just three polygons. If there is any map which requires five colors, there must be a standard map of that kind. Such a map with the smallest number of polygons is said to be minimal. It is known that a minimal map contains at least 32 polygons, of which at least 15 are pentagons while at least one has six, seven or eight sides. The author improves this last statement by showing that

$$3p_6+p_7 \geqq p_8+\sum_{i>8}(i-9)p_i+24,$$

where p_i is the number of i-gons. (Thus at least eight of the polygons have six or seven sides.) The middle 16 pages are occupied by a proof of the following lemma: In a minimal map there is no n-gon ($6 \leqq n \leqq 9$) touching $n-3$ successive pentagons. The method consists in reducing a given map A, in which an n-gon touches $n-3$ pentagons, to a simpler map B, such that the possibility of coloring B implies the possibility of coloring A. This requires separate consideration of about four hundred alternative colorings.

H. S. M. Coxeter (Toronto, Ont.).

Hudekoff, N. Über die allgemeine Lage von $n+2$ Punkten in R^n. Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 249–276 (1941). (Russian. German summary) [MF 4552]

The author investigates collections of $n+2$ points in general position in the Euclidean R^n. The collections $(p_0, \cdots, p_{n+1})$ and $(q_0, \cdots, q_{n+1})$ are called homeomorphic if their convex hulls can be mapped topologically on each other so that each m-simplex spanned by points p_i is carried into an m-simplex spanned by points q_i ($m=0, 1, \cdots, n$). The author subdivides the points of a collection $(p_0, \cdots, p_{n+1})$ into two non-vacuous classes as follows: p_i and p_k belong to the same class if they are separated by the $(n-1)$-manifold spanned by the n remaining p_i. The result of the paper is the statement that two collections (p_i) and (q_i) are homeomorphic if and only if they have similar subdivisions in classes. This result seems rather obvious to the reviewer, for it is easily seen that the subdivision in classes considered by the author coincides with the subdivision of the collection of $n+2$ non-vanishing real numbers c_i satisfying $\sum_{i=0}^{n+1} c_i p_i = 0$ and $\sum_{i=0}^{n+1} c_i = 0$ in two classes according to their signs, and from this it follows almost immediately that the simplexes $(p_0, \cdots, p_m)$ and $(p_{m+1}, \cdots, p_{n+1})$ have a point in common if and only if $p_0, \cdots, p_m$ are in one class and $p_{m+1}, \cdots, p_{n+1}$ in the other class. This shows the necessity of the condition and the sufficiency can be proved by equally elementary methods. The proof given by the author is long and complicated. He states in a remark preceding the paper that the proofs could be shortened, but some of the theorems on determinants used in the present proof are in his opinion of an independent interest.

W. Hurewicz (Chapel Hill, N. C.).

Greitzer, S. L. Combinatorial topology of polyhedra. J. Math. Phys. Mass. Inst. Tech. **19**, 260–288 (1940). [MF 3838]

A maximal tree on a polyhedron (reticulated 2-dimensional manifold) P is a 1-dimensional subcomplex which contains all the vertices. Given P and a maximal tree T, a unique symbolic expression $\omega(P, T)$, called a monomial, is derived. Every monomial ω determines uniquely a polyhedron $P(\omega)$ and furthermore $P(\omega(P, T)) = P$. Certain symbolic modifications of ω, called allowable transformations, are defined. It is possible to change ω to ω' by allowable transformations if and only if $P(\omega) = P(\omega')$. Application to the map coloring problem on various orientable and non-orientable surfaces is indicated.

R. H. Fox (Urbana, Ill.).

Robbins, Herbert. On the classification of the mappings of a 2-complex. Trans. Amer. Math. Soc. **49**, 308–324 (1941). [MF 3925]

The difficult problem of characterizing homotopy classes of mappings by combinatorial properties is solved in this paper for the case of the mappings of a 2-complex into an arbitrary space T. The necessary and sufficient condition for the homotopy of two mappings involves the use of chains of K with coefficients from the fundamental group of T and the 2-dimensional homotopy group of T.

W. Hurewicz (Chapel Hill, N. C.).

Eilenberg, Samuel. Continuous mappings of infinite polyhedra. Ann. of Math. (2) **42**, 459–468 (1941). [MF 4295]

This paper is based on the results established previously [Ann. of Math. (2) **41**, 231–251 (1940); these Rev. **1**, 222]

These had been formulated originally in terms of cohomologies and are translated into terms of homology in the present paper. One of the main results is the following generalization of the well-known theorem of H. Hopf on mappings of polyhedra into spheres: If K is a connected finite or infinite cell complex and Y an arcwise connected topological space such that the homotopy groups $\pi_i(Y) = \pi_i(K)$ vanish for $i=1, \cdots, n-1$ $(n \geqq 2)$ and in addition the cohomology groups of K of dimension i with coefficients from the group $\pi_i(Y)$ vanish for all dimensions $i>n$, then the homotopy classes of continuous mappings of K in Y are characterized by the homomorphisms of the n-dimensional homology group of K with integer coefficients into the corresponding group of Y induced by the mappings (the homology groups of Y are to be defined in terms of singular chains). In the case of a finite n-dimensional K this theorem is contained in a result of the reviewer [Nederl. Akad. Wetensch., Proc. 39, 117–126 (1936), in particular, p. 124].

The author gives several applications of his results: He establishes necessary and sufficient conditions that a finite or infinite polyhedron be of the same homotopy type as the n-sphere [this is a generalization of a result of the reviewer, l.c.] and discusses the configuration of a S^{r-n-1} topologically imbedded into S^r. The complement $S^r - S^{r-n-1}$ $(n \geqq 2)$ has the homotopy type of S^n if and only if it is simply connected (if $n=1$ this condition has to be replaced by the requirement that the group $\pi_1(S^r - S^{r-2})$ be infinitely cyclic and that $S^r - S^{r-s}$ be i-simple for all dimensions $i \geqq 2$). If this condition is satisfied every S^n topologically imbedded in the same S^r and having the linkage coefficient 1 with S^{r-n-1} is a deformation retract in $S^r - S^{r-n-1}$. *W. Hurewicz.*

Eilenberg, Samuel and MacLane, Saunders. Infinite cycles and homologies. Proc. Nat. Acad. Sci. U.S.A. 27, 535–539 (1941). [MF 5734]

Group extensions make their first appearance in homology theory. They furnish an efficient tool for studying the homology groups of an infinite locally finite complex K. All groups being Abelian, let Ext $\{G, H\}$ be the totality of classes of equivalent extensions of G with preassigned factor group H, converted into a group by means of a certain natural composition law. For topological G, H let Hom $\{G, H\}$ be the topologized group of continuous mappings $G \to H$; in particular, let Ch G = Hom $\{G, R\}$, where R is the group of real numbers mod 1. For discrete G, H let $G \cdot H$ = Ch Hom $\{H,$ Ch $G\}$ (a simple modification defines $G \cdot H$ also for compact 0-dimensional H). If H is finite, then Ext $\{G, H\} \cong G \cdot$ Ch H.

For given dimension q and coefficient group G, let $H^q(K, G)$ be the group Z^q of infinite cycles of K modulo the group B^q of boundaries of infinite chains and let $H_w^q(K, G)$ be Z^q modulo the weak boundaries B_w^q ($z \varepsilon B_w^q$ if for every finite L of K there is a chain z_1 in $K - L$ such that $z - z_1 \varepsilon B^q$). Also let $\mathfrak{H}^q(K, G)$ be the finite cocycles modulo the coboundaries of finite chains and let $\mathfrak{T}_q$ consist of those elements of $\mathfrak{H}_q$ which are of finite order. The authors announce without proof a number of homomorphic relations involving these various groups and the operations Hom, Ch, Ext. Corollaries include the direct sum decompositions:

$$H^q(K, G) \cong \text{Hom}\{\mathfrak{H}_q(K, I), G\} + \text{Ext}\{G, \mathfrak{H}_{q+1}(K, I)\},$$
$$H_w^q(K, G) \cong \text{Hom}\{\mathfrak{H}_q(K, I), G\} + G \cdot \text{Ch}\, \mathfrak{T}_{q+1}(K, I),$$

where I is the group of integers. For compact G,

$$G \cdot \text{Ch}\, \mathfrak{T}_{q+1}(K, I) \cong \text{Hom}\{G, \mathfrak{T}_{q+1}(K, I)\}.$$

Let K be a fundamental infinite complex of a compact metric space X. Application of the preceding results shows that the Čech homology group $\mathfrak{H}^q(X, G)$ is determined by G and the (Čech cohomology) groups $\mathfrak{H}_q(X, I)$ and $\mathfrak{T}_{q+1}(X, I)$. This is essentially Steenrod's universal coefficient group theorem for compact metric spaces. *P. A. Smith.*

Whitehead, J. H. C. On incidence matrices, nuclei and homotopy types. Ann. of Math. (2) 42, 1197–1239 (1941). [MF 5820]

It has been established [J.H.C. Whitehead, Proc. London Math. Soc. (2) 45, 243–327 (1939); in particular, p. 278] that two complexes belong to the same homotopy type if and only if they are deformation retracts of a third complex in which they are both imbedded. If the retracting deformation takes place a simplex at a time the two complexes are said by the author to belong to the same nucleus. Homotopy type is a topological invariant, but topological invariance of the nucleus is unproved. The author gives examples of complexes which belong to the same homotopy type but not to the same nucleus.

The principal object is to make a contribution to the theory of lens spaces. The classical result [Seifert and Threlfall, Lehrbuch der Topologie, Leipzig, 1934, p. 279] is that lens spaces (m, q) and (m, r) are topologically equivalent only if $r \equiv \pm t^2 q \pmod{m}$ for some t. The author proves that (m, q) and (m, r) belong to the same homotopy type if and only if $r \equiv \pm t^2 q \pmod{m}$. This result implies, incidentally, a negative answer to questions Q. 1, $\cdots$, 4 [J.H.C. Whitehead, loc. cit., §10] and also contradicts a previously published (false) theorem [J. H. C. Whitehead, Quart. J. Math., Oxford Ser. 10, 81–83 (1939)]. (The nature of the error in the earlier paper is discussed.)

The bulk of the paper is devoted to the machinery by which the main theorem is proved. That this machinery may be of importance in itself is indicated by the profundity of the author's result on lens spaces. Following a procedure of Reidemeister (which involves a more or less arbitrary choice of a tree in K^n and a fundamental domain in the universal covering complex $\bar{K}^n$) the author defines, for any n-dimensional complex K^n, incidence numbers which are elements of the integral ring of the fundamental group $\pi_1(K^n)$ of K^n. Using these incidence numbers, incidence matrices $r^n, \cdots, r^1$ are defined in the usual way. A natural system (r, R) consists of the matrices $r^n, \cdots, r^3$ together with a certain system R of generators and relations for $\pi_1(K^n)$ which is uniquely determined by Reidemeister's "tree." Natural systems can be defined algebraically and can be algebraically classified according to two kinds of equivalence, L-equivalence and L^*-equivalence. Theorem 8 states that two complexes belong to the same nucleus only if their natural systems are L-equivalent, and to the same homotopy type only if their natural systems are L^*-equivalent. The converse of theorem 8 holds when the complexes are 3-dimensional (or when the dimension n of the complexes is greater than 3 and their higher homotopy groups of dimensions less than $n-1$ vanish). *R. H. Fox.*

Chogoshvili, George. On the homology theory of topological spaces. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 337–342 (1940). (English. Georgian summary) [MF 5277]

A proof is offered that the Čech and Kolmogoroff homology theories in a T_3-space are equivalent whenever

a bicompact coefficient group is used. The proof assumes that, for any two finite closed coverings K_α, K_β such that K_α is a refinement of K_β, a unique projection ρ_β^α of K_α into K_β is given, and these projections satisfy the transitivity condition $\rho_\gamma^\beta \rho_\beta^\alpha = \rho_\gamma^\alpha$. It has not been proved that such a selection is possible for the family of all closed coverings. Since the complete subfamily of Kurosch coverings [Compositio Math. 2, 471–476 (1935)] has the desired property, it appears likely that the proof may be remedied.

N. E. Steenrod (Chicago, Ill.).

Alexandroff, Paul. Homologie-Gruppen allgemeiner Projektionsspektra. Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 213–219 (1941). (Russian. German summary) [MF 5298]

Following Čech the homology theory of a space can be described using (1) family $\{K_\alpha\}$ of the nerves of finite open coverings of the space, (2) the projections that connect up the nerves of a covering and of its refinement. Leaving the space completely out of the picture, Alexandroff obtains the concept of a projection spectrum for which he outlines a homology and cohomology theory, using implicitly the familiar method of direct and inverse systems of groups. The projection spectra as defined by the author are almost identical with the nets of S. Lefschetz [Lectures on Topology, University of Michigan Press, Ann Arbor, Michigan, 1941, pp. 1–28, in particular, p. 26; these Rev. 3, 132].

S. Eilenberg (Ann Arbor, Mich.).

Samelson, Hans. Beiträge zur Topologie der Gruppen-Mannigfaltigkeiten. Ann. of Math. (2) 42, 1091–1137 (1941). [MF 5818]

In this paper, following up the line of investigation which was recently started by H. Hopf [Ann. of Math. (2) 42, 22–52 (1941); these Rev. 3, 61], some notable progress is made in the study of the topological structure of compact Lie-groups and of the corresponding homogeneous spaces. As in Hopf's paper, the methods are those of "classical" topology, combined with an extensive use of Hopf's inverse homomorphism, and with some of the simpler results concerning fibre-spaces. No use is made of the special properties of the "classical" simple Lie-groups. The domain of coefficients which is used throughout for the definition of chains and cycles is that of rational numbers; as mentioned in the introduction, the practically equivalent choice of the real numbers as coefficients would have insured the isomorphism of the Betti-ring with the ring of differential forms, and would have made it possible to use the powerful methods of de Rham, Cartan and Hodge, which seem particularly well adapted to such problems; it is to be hoped that this will be done later by the author.

Let G be a (connected) compact Lie-group of dimension n; amongst the elements (homology-classes) of the Betti-groups of all dimensions d ($0 \leqq d \leqq n$) in G, two bilinear operations (or "multiplications") are introduced. One is the intersection, denoted by $x \cdot y$, making the direct sum of all Betti-groups into the usual Betti-ring; if $d(x)$ is the dimension of a homology-class x, and $\delta(x) = n - d(x)$, then $\delta(x \cdot y) = \delta(x) + \delta(y)$. On the other hand, if points p, q in G describe cycles belonging to the homology-classes x, y, respectively, then $p \cdot q$ (the product of p and q by the group operation) describes a cycle, the homology-class of which is called the Pontrjagin product of x and y, and denoted by $x \circ y$; and $d(x \circ y) = d(x) + d(y)$; this makes the direct product of all Betti-groups into a ring, called by the author the Pontrjagin ring of G. The Betti-groups, with this twofold law of multiplication, are investigated, and the structure of that system completely determined, in chapter I (the same had been done by H. Hopf for the Betti-ring); the result is expressed as an isomorphism between that system and the system of all Betti-groups of a suitable product $S_{m_1} \times S_{m_2} \times \cdots \times S_{m_l}$ of spheres S_{m_i} of odd dimensions m_i (l being the rank of G), when two products amongst the latter are defined by the intersection and by the operation of direct product (and suitable algebraic conventions), respectively. It turns out that there is a sort of duality between the two products.

As a first application (chap. II, §1), it is shown that a homomorphism of G into a group G' induces a homomorphism of the Pontrjagin ring of G into that of G', the "kernel" of which (ideal of elements which are mapped onto 0) is discussed; this enables the author to determine the homology type of the mapping $(p \to p^2)$ of G into itself. The case where the homomorphism of G into G' reduces to an isomorphism of G into G' is considered, and this leads to conclusions about the homology-groups of a subgroup of a given group. For instance, if $G_1 \supset G_2 \supset \cdots \supset G_n$ is a decreasing sequence of groups, and G_{i+1} not ~ 0 in G_i, then G_n not ~ 0 in G_1.

The rest of the paper is devoted to homogeneous spaces, that is, to spaces W with a compact transitive Lie-group G of automorphisms; such a space can also be defined as the space G/U of the cosets determined in G by a (closed) subgroup U; the cosets may be considered as fibres in G, making G into a fibre-space, with the base-space G/U; a useful tool here is Feldbau's theorem (a fibre-space, with a topological element as base-space, is a direct product of the latter by the fibre). One may then try to derive properties of G/U from properties of G and U, or conversely; the author adopts the latter point of view in chapter III, the former in chapter IV. In chapter III, $W = G/U$ is taken to be a sphere S_m (of dimension m); it is shown that, for m odd, the Betti-ring of G is isomorphic to that of $U \times S_m$, and that U not ~ 0 in G; for m even, the Betti-groups of G and U are respectively isomorphic to those of $\Pi \times S_{2m-1}$ and of $\Pi \times S_{m-1}$, where Π is a product of spheres of odd dimensions; and $U \sim 0$ in G. This contains the known results on the homology-rings of the "classical" simple Lie-groups of the four classes A, B, C, D (that is, the orthogonal, unitary and symplectic groups). Finally, the main result of chapter IV concerns the case where G and U are subject to the sole condition that U not ~ 0 in G: the Betti-ring of $W = G/U$ is then isomorphic to that of a product of spheres of odd dimensions, and the Betti-ring of G is isomorphic to that of the product $W \times U$.

A. Weil (Haverford, Pa.).

Pontrjagin, L. Über die topologische Struktur der Lieschen Gruppen. Comment. Math. Helv. 13, 277–283 (1941).

The homology-ring (rational coefficients) of a closed group manifold G is isomorphic with that of a topological product Π_G of certain spheres S_i of odd dimensions i [cf. L. Pontrjagin, C. R. Acad. Sci. Paris 200, 1277–1280 (1935); H. Hopf, Ann. of Math. (2) 42, 22–52 (1941); cf. these Rev. 3, 61]. So the question arises whether G and Π_G are actually homeomorphic. This is here answered negatively for the groups A_n of unitary unimodular matrices of order $n+1$, the corresponding Π_{A_n} of which is $S_3 \times S_5 \times \cdots \times S_{2n+1}$; in fact, it is proved that A_n is not homeomorphic with the product $S_3 \times M$, where M is any topological space. This is done by showing that every mapping of a 4-sphere into A_n

is homotopic to 0, whereas it is known that there exists one (and only one) essential class of mappings of a 4-sphere onto a 3-sphere, and so at least one essential mapping of a 4-sphere into $S_3 \times M$. Using the decomposition of A_k into cosets of A_{k-1} (considered in the obvious way as a subgroup of A_k), with S_{2k+1} as coset space, it is easily shown that every mapping of a 4-sphere into A_n can be so deformed as to lie finally on the subgroup A_1 which is homeomorphic with S_3. So it is sufficient to show that the (there is only one) essential mapping of S_4 onto A_1 is homotopic to 0 in A_2. This is now done by means of explicit formulae.

H. Samelson (Princeton, N. J.).

Steenrod, Norman E. Topological methods for the construction of tensor functions. Ann. of Math. (2) **43**, 116–131 (1942). [MF 6065]

Let M be a differentiable manifold. The author considers the space M' of tensor functions, or ordered sets of vectors, etc., of a given kind defined over M; this is a fibre bundle [Whitney, Proc. Nat. Acad. Sci. U. S. A. **26**, 148–153 (1940); these Rev. **1**, 220]. The fibre F_P over the point P is the set of values of tensors, etc., at P. Set $\Pi(F_P)=P$. A certain subspace M'' of M' is considered, for instance, the space of non-vanishing tensors. The problem is to choose for each $P \in M$ a point $f(P) \in F_P$ so that f is differentiable. If the homotopy groups of a fibre vanish for dimensions less than h, it is easy to construct f (to be continuous) over the h-dimensional part M^h of M. One can extend f over M^{h+1} if and only if the "characteristic cocycle" $c^{h+1} \sim 0$ as in the work of Stiefel and the reviewer [see, for instance, Whitney, loc. cit.]. If $h=1$, it is assumed that the fundamental group is Abelian. In general, one must use "local coefficient groups" in c^{h+1}. It is noted that this concept is equivalent to the "Überdeckungen" of Reidemeister [Topologie der Polyeder, Leipzig, 1938]. If f is continuous, it can be approximated by a differentiable function. As an application, a simple proof of the existence of a Riemann metric in M is given. The theory is extended to cover the "fibre spaces" of Hurewicz and the author [Proc. Nat. Acad. Sci. U. S. A. **27**, 60–64 (1941); these Rev. **2**, 323].

H. Whitney.

ANALYSIS

***Courant, Richard and Robbins, Herbert. What Is Mathematics?** Oxford University Press, New York, 1941. xix+521 pp. $3.75.

The book is an elementary approach to modern mathematics in which processes of thought, rather than mechanical routines, are emphasized, and is a model of lucid exposition. From a level approximately that of a sound high-school training, the development proceeds by direct paths to some of the best content of mathematics; and fundamental ideas are made strikingly clear by well-chosen, simple examples. The text is illustrated with nearly 300 diagrams, and much of the clarity arises from a generous use of geometrical interpretation.

The book begins with an elementary discussion of the natural numbers, number theory and the number system, including such topics as the distribution of primes, Fermat's last theorem, Euclid's algorithm and irrational, complex, algebraic and transcendental numbers. The next sections are concerned with geometry. There is an outstanding chapter on geometrical constructions, and there are two chapters on projective and non-Euclidean geometry and topology which are excellent introductions to these subjects. A chapter on functions and limits then prepares the ground for the final sections of the book, a discussion of maxima and minima followed by an introduction to the calculus. The chapter on maxima and minima is rich in problems which lend themselves to direct treatment.

Although it is inevitable in a work of this scope that there should be a few historical statements with which specialists may disagree, it is nevertheless a work of extraordinary perfection. *D. C. Spencer* (Cambridge, Mass.).

***Picone, Mauro. Appunti d'analisi superiore.** Editore Alfredo Rondinella, Napoli, 1940. 848 pp. L. 110.

Table of contents: Ch. I. Elementi della teoria delle funzioni analitiche. Ch. II. Nozioni fondamentali sulle funzioni armoniche. Ch. III. Proprietà fondamentali delle serie di Fourier. Ch. IV. Funzioni e serie di Laplace. Ch. V. Complementi alla teoria delle funzioni armoniche. Ch. VI. Serie doppie di Fourier e di Legendre. Ch. VII. Su alcune equazioni classiche della fisica-matematica.

Schaeffer, A. C. Oscillations of the derivatives of a function. Proc. Nat. Acad. Sci. U. S. A. **28**, 62–64 (1942). [MF 6139]

If, in an interval of length $2L$ and center a, the function $f(x)$ is real valued and possesses an nth derivative, if $|f(x)| \leqq M$ and $f^{(n)}(x)$ changes sign less than $n-1$ times ($n \geqq 2$), then $|f'(a)| < (20n)^{3n} M/L$. The author indicates several consequences of this remarkable new inequality whose detailed proof is not given. The following theorem, easily derived from the inequality, generalizes a result of Serge Bernstein and answers a question raised by the reviewer. A real valued function, possessing derivatives of all orders in an open interval, is necessarily analytic in the interval if none of its derivatives changes sign more than k times, where k is a fixed number, the same for all derivatives. [S. Bernstein had the case $k=0$.]

G. Pólya.

Bosanquet, L. S. A mean value theorem. J. London Math. Soc. **16**, 146–148 (1941). [MF 6009]

M. Riesz and others have shown that, if $0 < \alpha \leqq 1$, $0 \leqq y \leqq x$ and $f(t)$ is Lebesgue integrable, then

$$\left| \int_0^y (x-t)^{\alpha-1} f(t)\,dt \right| \leqq \text{m.u.b.}_{0 \leqq u \leqq y} \left| \int_0^u (u-t)^{\alpha-1} f(t)\,dt \right|,$$

where m.u.b. denotes the least measurable upper bound. This note obtains an analogue in which integrals are replaced by sums. Let $A_\nu^\sigma = (\sigma+\nu)!/\sigma!\nu!$. If $0 \leqq \alpha \leqq 1$ and $0 \leqq m \leqq n$, then

$$\left| \sum_{\nu=0}^{m} A_{n-\nu}^{\alpha-1} a_\nu \right| \leqq \max_{0 \leqq \mu \leqq m} \left| \sum_{\nu=0}^{\mu} A_{\mu-\nu}^{\alpha-1} a_\nu \right|$$

for each sequence $a_1, a_2, \cdots, a_m$ of real or complex numbers. Examples are given to show that the last estimate can fail when $\alpha > 1$ and when $\alpha < 0$.

R. P. Agnew.

Martinotti, Pietro. Di alcune recenti medie. Pont. Acad. Sci. Acta **5**, 113–121 (1941). [MF 5889]

General remarks on the concept of mean value concluded by two unusual examples of means arising in elementary statics and optics, respectively.

I. J. Schoenberg.

Calculus

***Curtiss, D. R. Maxima and minima of functions of two or more variables.** Northwestern University Studies in Mathematics and the Physical Sciences, no. 1: Mathematical Monographs, vol. 1, pp. 1–43. Graduate School, Northwestern University, Evanston, Ill., 1941. $2.25.

Various methods for determining maxima and minima for functions of two or more variables are given, especially those of Scheefer, Stoltz and von Dantscher. The monograph deals for the most part with proper maxima and minima of functions of two variables. Procedures to be adopted for functions of more than two variables are indicated. The problem is also studied from the following original point of view: If $F(x, y)$ is a polynomial in y it may be written: (1) $F(x, y)=a_0(x)y^n+a_1(x)y^{n-1}+\cdots+a_n(x)$. The theorems of Sturm and of Budan-Fourier are then applied to get information about the real roots of $F(x_1, y)=0$ for x_1 small, and thus determine how many branches of $F(x, y)$ are real when x is small. If $a_n(0)=0$ then $F(x, y)$ vanishes at the origin and information may be obtained about the real branches of $F(x, y)=0$ which pass through the origin, if there are such branches. In special cases (2) $a_0(x)=\text{const.}\neq 0$, $a_1(0)=a_2(0)=\cdots=a_n(0)=0$. Hence every solution y of (1) tends to zero as x tends to zero, and all real solutions $y=\varphi(x)$ for sufficiently small x pass through the origin. If $F(x, y)$ is in the class for which (1) and (2) hold, and if for x sufficiently small $F(x, y)=0$ has no real solutions $y=\varphi(x)$, then $F(x, y)$ has an extremum at the origin.

R. L. Jeffery (Wolfville, N. S.).

Roussel, André. Sur la définition des fonctions continues par leur accroissement infinitésimal. C. R. Acad. Sci. Paris **212**, 593–595 (1941). [MF 4921]

If $f(x)$, $g(x, h)$ and $\omega(h)$ are such that $f(x+h)-f(x)=g(x, h)+\epsilon\omega(h)$ (f and g are continuous, $g(x, 0)\equiv 0$, $\omega(h)$ fixed, tending to 0 with h), then $g(x, h)$ is called the "principal part to within $\epsilon\omega(h)$" of $f(x)$. This note is concerned with conditions on g under which, for a given function ω, there exists a function $f(x)$ of which g is the principal part with respect to $\epsilon\omega$. A sufficient condition is stated, without proof, for the general case; with the additional hypothesis that g possesses partial derivatives, except perhaps for $h=0$, a construction for $g(x, h)$ is indicated. *A. Dresden.*

Vicente Gonçalves, J. Sur l'inconnue θ du théorème des accroissements finis. Portugaliae Math. **2**, 121–138 (1941). [MF 4576]

The most interesting result in this paper is that, if $f(x)$ possesses derivatives of sufficiently high order at $x=a$ and if $f''(a)=\cdots=f^{(m-1)}(a)=0$, $f^{(m)}(a)\neq 0$, then the "mean value" $\theta(a, h)$, which occurs in the law of the mean, has the property that $\lim_{h\to 0}\theta(a, h)=m^{1/(1-m)}$. The converse, although not explicitly stated, is implied throughout; it follows readily from the direct theorem together with the fact that, for real numbers p and q, the equality $p^{q-1}=q^{p-1}$ implies $p=q$. The definition of the function $\theta(x, h)$ is completed by setting $\theta(x, 0)=\lim_{h\to 0}\theta(x, h)$. A number of special questions are now considered, the cases in which θ is independent of x, of h, or of both x and h (the latter happens only for quadratic polynomials). Next comes the uniqueness of θ as a function of h only and the calculation of its derivatives with respect to h, under the hypothesis $f''(a)\neq 0$, with explicit formulas for the first and second derivatives. In the last section, the author considers the case $f''(a)=0$, deals briefly with the mean value occurring in the extended law of the mean and gives some indications for a treatment of the corresponding problems for functions of two variables. The paper has to be read with some precautions, since there are a number of misprints as well as some loosely worded statements. The discussion of the uniqueness of θ on pp. 126 and 127 seems unnecessarily complicated; the calculations on p. 132 are obscured by the double use of λ and by the incorrect statement that $\lim\lambda=-3/4!8$. It should be worth while to determine the connections of this paper with Aumann's papers on general mean values. *A. Dresden* (Swarthmore, Pa.).

Picone, M. Sul calcolo delle derivate d'ordine superiore. Period. Mat. (4) **21**, 141–150 (1941). [MF 5385]

Sufficient conditions are given for the simultaneous limit of the function

$$f(x_1, \cdots, x_{n+1})\equiv\sum_1^{n+1}f(x_i)/(x_i-x_1)\cdots(x_i-x_{n+1})$$

(with zero factor omitted) to be equal to $f^{(n)}(x)/n!$ as all the points $x_1, \cdots, x_{n+1}$ approach x. The order of the infinitesimal difference $f^{(n)}(x)/n!-f(x_1, \cdots, x_{n+1})$ is studied for both real and complex domains. In the latter case, if $f(x)$ is analytic, this order is 1 in general, but may be as high as $n+1$ if the points $x_1, \cdots, x_{n+1}$ are vertices of a regular polygon with $n+1$ sides and center at x.

P. W. Ketchum (Urbana, Ill.).

Kimball, W. S. Partial derivatives of derivatives. II. Philos. Mag. (7) **32**, 137–154 (1941). [MF 5318]

[The first part appeared in Philos. Mag. (7) **30**, 190–222 (1940); cf. these Rev. **2**, 78.] The author continues the discussion of the partial derivatives of a line integral. In the present paper he treats the case of an arbitrary path, without indication of any restrictions. The parallel displacement of a portion of the path is determined by a vector (h, k), the new path consisting of fixed parts of the original path, of 2 straight line segments and of the displaced part. He considers the derivatives of this integral with respect to the components h and k of the vector, at $h=k=0$, keeping the endpoints of the displaced part fixed. In spite of these further developments of the ideas, the reviewer remains unregenerate as to the significance of the new definition of derivative.

A. Dresden (Swarthmore, Pa.).

Freudenthal, Hans. Eine Restabschätzung bei der Taylorschen Formel und ihre Anwendung auf die logarithmische und die binomische Reihe. Nieuw Arch. Wiskde (2) **20**, 269–272 (1940). [MF 5210]

Jacobsthal, E. et Tambs Lyche, R. Sur la démonstration de quelques théorèmes classiques du calcul différentiel. Arch. Math. Naturvid. **44**, 73–91 (1941). [MF 4425]

Rados, Gusztáv. Eine neue Herleitung der Gram'schen Kriterien. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 7–18 (1940). (Hungarian. German summary) [MF 4444]

Hemmingsen, Erik. On Weierstrass sums for integrals involving second derivatives. Rep. Math. Colloquium (2) **3**, 31–33 (1941). [MF 5779]

The method of Menger [Proc. Nat. Acad. Sci. U. S. A. **26**, 660–664 (1940); these Rev. **2**, 78] for proving Green's formula by means of Weierstrass sums is now extended to

formulas involving second derivatives, such as

$$(1) \qquad \int\int (v_{xx}+v_{yy})dxdy=\int(-v_x dy+v_y dx).$$

The finite identity, of which (1) is shown to be the limit under the hypothesis that the Riemann integrals exist, involves second difference quotients. *E. F. Beckenbach.*

Artin, Emil. On the independence of line integrals on the path. Proc. Nat. Acad. Sci. U.S.A. **27**, 489–490 (1941). [MF 5452]

The author states the following theorem. Let dA be a differential in two variables x, y. In a simply connected domain the integral $\int_C dA$ will be equal to zero (for the boundary of any triangle) if and only if dA is regular at all points of the domain, that is, if for any sequence Δ_n of similar triangles with areas a_n tending to zero we have $\lim (1/a_n)\int_{\Delta_n} dA=0$. The author deduces some corollaries for the differentials $dA=P(x, y)dx+Q(x, y)dy$, by supposing that P, Q have differentials in the sense of Stoltz.

G. Fubini (Princeton, N. J.).

Erim, Kerim. Über die Darstellung mehrfacher Integrale. Rev. Fac. Sci. Univ. Istanbul (A) **5**, 191–214 (1941). (German. Turkish summary) [MF 5425]

Let B be a simply connected closed plane region bounded by a curve Γ with polar equation $r=R(\varphi)$, in which $R(\varphi)$ is continuous and single valued. Let f denote a continuous point function in B and f', f'', etc., the successive partial derivatives of f with respect to r. The author shows that quantities $c_0, c_1, \cdots, c_m$ can be found, dependent on B but independent of f, such that

$$\int\int_B f dxdy=c_0\int_\Gamma f ds+\int_\Gamma c_1 f' ds+\int_\Gamma c_2 f'' ds/2!+\cdots+\int_\Gamma c_m f^{(m)} ds/m!+R_{m+1}.$$

In the integral $\int_\Gamma c_n f^{(n)} ds/n!$ $(n=1, 2, \cdots, m)$ the boundary Γ is divided into $n+1$ (or more) segments on each of which c_n is a constant.

If B is a volume bounded by a closed surface Ω with equation $r=R(\varphi, \theta)$ in spherical coordinates, where R is single valued, the corresponding formula is

$$\int\int\int_B f dxdydz=D\int\int_\Omega f d\omega+\int\int_\Omega D_1 f' d\omega+\cdots+\int\int_\Omega D_m f^{(m)} d\omega+R_{m+1}.$$

Here in the integral $\int\int_\Omega D_n f^{(n)} d\omega$ $(n=1, 2, \cdots, m)$ the surface Ω is divided into $(n+1)(n+2)/2$ (or more) subregions on each of which D_n is constant. The author gives formulas for the calculation of the c's and D's and considers the magnitude of the remainders R_{m+1}. *W. E. Milne.*

Sastry, B. S. Note on centre of gravity. J. Mysore Univ. Sect. B. **1**, 105–107 (1941). [MF 5138]

The author gives formulas for the centroid of solid figures bounded by a lateral surface and two parallel intercepting planes (not excluding the degenerate cases of tangent planes) in terms of the areas of the extreme sections and of the mean section. The solid figure is in a three dimensional Euclidean space and the lateral surfaces considered are general enough to include lateral boundaries consisting of parts of a finite number of ruled surfaces.

A. D. Michal (Pasadena, Calif.).

Sastry, H. Sreenivasa. On the function defined by the relation $\phi_r(x)=x\phi'_{r-1}(x)$. Math. Student **8**, 127–130 (1940). [MF 4316]

If $\phi_0(x)$ is given and $\phi_r(x)$, $r\geqq 1$, is defined by $\phi_r(x)=x\phi'_{r-1}(x)$, then $\phi_r(x)$ can be expressed linearly in terms of $\phi_0', \cdots, \phi_0^{(r)}$. The author investigates the coefficients of this expression. *O. Szász* (Cincinnati, Ohio).

Kesava Menon, P. A generalisation of the circular and hyperbolic functions. Math. Student **8**, 112–117 (1940). [MF 4313]

The author investigates functions of the form

$$P_r(x)=1/n\sum_{s=0}^{n-1} t^{-rs}\exp(m_s x),$$

$r=0, 1, \cdots, n-1$, where $t=\exp(2\pi i/n)$. They satisfy a certain determinant relation; an addition formula is established, and particular cases for the constants m_s are considered. *O. Szász* (Cincinnati, Ohio).

Opatowski, I. Bactericidal irradiation of air. J. Franklin Inst. **230**, 643–645, discussion, 645 (1940). [MF 4460]

It is shown that the integrals $\int_0^x r^2 e^{-1/r} dr$ and $\int_0^x t e^{-1/x^2} dx$ can be computed by means of the error integral and the exponential integral, respectively. The integrals occur in a paper by Wells on bactericidal irradiation of air [same J. **229**, 347–372 (1940)]. *W. Feller* (Providence, R. I.).

Schubert, Gerhard. Bemerkungen zu einigen bestimmten Integralen. Z. Angew. Math. Mech. **21**, 190 (1941).

Corrections to a few formulas in Bierens de Haan: Nouvelles tables d'intégrales définies, Leyden, 1867.

Bongiovanni, Emilia. Sopra alcuni notevoli integrali doppi. Boll. Un. Mat. Ital. (2) **3**, 296–299 (1941). [MF 5600]

Riabouchinsky, Dimitri. Les diviseurs de zéro et le concept de l'origine d'un nombre. C. R. Acad. Sci. Paris **212**, 677–680 (1941). [MF 4926]

The author proposes the development in systematic fashion of a theory of functions of complex variables $x+jy$, and also of $x+ky$, where, as introduced by Grossmann, $(j+1)(j-1)=0$, $j\neq\pm 1$, and $k^2=0$, $k\neq 0$. To this end he discusses multivalued operations of returning from the absolute value, and of returning from the limit, as suggested by the classical "determination of the true values of the indeterminate forms $0\cdot\infty$, $\infty:\infty$, $\infty-\infty$, etc."

A. A. Bennett (Providence, R. I.).

Chazy, Jean. Sur la formule du double produit vectoriel. C. R. Acad. Sci. Paris **211**, 449–450 (1940). [MF 5358]

A proof of the formula of ordinary vector analysis:

$$(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}.$$

D. J. Struik (Cambridge, Mass.).

Tacchella, Giuseppe. Sulla rappresentazione analitica di figure composte. Atti Soc. Sci. Genova **5**, 254–266 (1940). [MF 5438]

Analytical representation of a few regular polygons and other rectilinear figures by means of elementary functions,

using $|x|$ in particular. For previous literature cf. Oakley [Amer. Math. Monthly **42**, 476–487 (1935); Bull. Sci. École Polytéch. Timişoara **7**, 191–199 (1937)].

W. Feller (Providence, R. I.).

Sbrana, Francesco. Sopra certe proprietà delle curve. Boll. Un. Mat. Ital. (2) **3**, 359–363 (1941). [MF 5701]

Roussel, André. Sur une extension simple de la notion de contact. C. R. Acad. Sci. Paris **212**, 527–528 (1941). [MF 4919]

The idea of an envelope of a one-parameter family of continuous curves is generalized by weakening the order of contact. The author then states that these generalized envelopes may exist even if the members of the family of curves do not possess tangents. *F. G. Dressel.*

Sergescu, P. Note on stationary osculatory curves. Pozitiva Revistă Mat. **1**, no. 9–10, 7 pp. (1941). (Rumanian) [MF 5691]

Si les courbes d'une famille F, osculatrices à une courbe C, sont stationnaires dans tous les points de C, cette courbe C est soit une courbe particulière F, soit une intégrale singulière de l'équation différentielle des courbes F. Résultats analogues pour les surfaces osculatrices stationnaires.

Author's summary.

Dolaptschiew, Bl. Eine Berührungstransformation in der Geometrie. Anwendung. (Knotenparabeln) Annuaire Univ. Sofia. Fac. Sci. Livre 1, Math. Phys. **36**, 327–367 (1940). (Bulgarian. German summary) [MF 3354]

The author studies some transformations and some curves in connection with the problem of determining the locus of centers of circles which pass through a given point and are tangent to a given curve of the plane. *W. Feller.*

Callaghan, Mary Patricia. Generalized Frégier curves. Revista Ci., Lima **43**, 105–109 (1941). [MF 4843]

That a variable chord of a conic (C) subtending a given angle at a fixed point M of (C) envelopes a conic (K) was first proved by Poncelet [Ann. Math. Pures Appl. **8**, 70 (1817–18); cf. also his Traité des propriétés projectives des figures, 1822, p. 280]. The author proves analytically that, as the point M describes the conic (C), the conic (K) envelopes a conic (L). She also considers the analogous problem when the two straight lines through M make supplementary angles with a fixed line in the plane. Frégier originally stated his theorem, both for the conic and the quadric, in Hachette's Correspondance sur l'École Polytech. **3**, 394 (1814–1816). *N. A. Court* (Norman, Okla.).

Theory of Series

Schwatt, I. J. An application of the bracketed number to the summation of a certain type of series. J. Math. Pures Appl. (9) **20**, 23–34 (1941). [MF 4638]

Evaluation of some finite trigonometric sums, based on a simple principle. *O. Szász* (Cincinnati, Ohio.).

Ghermănescu, M. On a development in series. Revista Mat. Timişoara **21**, 65–67 (1941). (Rumanian) [MF 5248]

The coefficients of the expansion

$$(1+ax)^{1/a}=\sum_0^\infty A_n(a)x^n$$

are found to be of the form

$$A_n(a)=e^a\sum_{\nu=0}^{n-1}c_{\nu n}a^{2n-\nu},$$

with rational $c_{\nu n}$. The author discusses two methods for their explicit determination. One method furnishes $c_{\nu n}$ for general n but only for successive values of ν; the other gives $c_{\nu n}$ for general ν and successive values of n.

I. J. Schoenberg (Philadelphia, Pa.).

Kosambi, D. D. A note on frequency distribution in series. Math. Student **8**, 151–155 (1940). [MF 5001]

Hill, J. D. Some theorems on subseries. Bull. Amer. Math. Soc. **48**, 103–108 (1942). [MF 6186]

Let $\sum u_k$ be an absolutely convergent series of real or complex terms, and let $s=\sum u_k$. To each ξ in the interval $0<\xi\leqq 1$ corresponds a diadic expansion $.\alpha_1\alpha_2\alpha_3\cdots$ in which each α_k is 0 or 1 and $\alpha_k=1$ for an infinite set of k; thus to each ξ corresponds a subseries $\sum\alpha_k u_k$ of $\sum u_k$ containing an infinite set of the terms of $\sum u_k$. The series $\sum\alpha_k u_k$ converges, for each ξ, to a function $\varphi(\xi)$. It is shown that the Riemann integral over $0\leqq\xi\leqq 1$ of $\varphi(\xi)$ is $s/2$ and hence that the "average value" of the subseries of $\sum u_k$ is $s/2$. Another proof could be given in terms of Rademacher functions. If $\sum v_k$ is a real convergent series which is not absolutely convergent and $\varphi_n(\xi)$ is defined for $n=1, 2, \cdots$ and $0<\xi\leqq 1$ as the sum of n terms of the series $\sum\alpha_k v_k$, then for all points except those in a set of the first category we have $\liminf \varphi_n(\xi)=-\infty$, $\limsup \varphi_n(\xi)=+\infty$. Let D be the complete metric space in which an element is an increasing sequence k_j of positive integers and the metric is that of Fréchet. Let f_n be the sum of n terms of the subseries $\sum v(k_j)$. Then for all sequences k_j in D except those in a set W of the first category we have $\liminf f_n=-\infty$, $\limsup f_n=+\infty$. *R. P. Agnew* (Ithaca, N. Y.).

Carlitz, L. The coefficients of the reciprocal of a series. Duke Math. J. **8**, 689–700 (1941). [MF 5953]

An infinite series of the form $f(u)=\sum c_m u^m/m!$, where the c_m are integers, is known as a Hurwitz series. If $c_0=0$, let $\lambda(u)$ be the function inverse to $f(u)$; it is easy to see that $\lambda(u)$ is expressible as a Hurwitz series also. The author makes the hypothesis that $\lambda(u)=\sum\epsilon_m u^m/m$ (ϵ_m integral) and studies the coefficients in the expansion of $u/f(u)$, which he writes in the form $\sum\beta_m u^m/m!$ (β_m rational). For example, if $f(u)=e^u-1$, the various conditions are satisfied and the numbers β_m become the Bernoulli numbers. It is proved that a theorem of the von Staudt-Clausen type holds for β_m, indeed, if m is even, $\beta_m\equiv-\sum_{p-1|m}p^{m/(p-1)}/p \pmod 1$; if m is odd, β_m is an integer or one-half of an integer. A similar result is derived for the expansion of $u^2/f^2(u)$. The proofs are elementary although certain simple properties of Hurwitz series are used.

Hurwitz had proved just such a theorem for the expansion of $\wp(u)$ in the lemniscatic case [Math. Ann. **51**, 196–226 (1899)=Hurwitz, Mathematische Werke, v. II, pp. 342–373]; the complex multiplication of $\wp(u)$ was used in this

proof. Hurwitz's results and others are derived as an application of the general theorems in this paper.

H. W. Brinkmann (Swarthmore, Pa.).

Rajagopal, C. T. On the rearrangement of conditionally convergent series. Ann. of Math. (2) **42**, 604–613 (1941). [MF 4960]

In this paper several theorems are proved relative to the behavior of the partial sum of a rearranged conditionally convergent series. The theorems are closely related to theorems of Pringsheim. *T. Fort* (Bethlehem, Pa.).

Rajagopal, C. T. Postscript to "Convergence theorems for series of positive terms." J. Indian Math. Soc. (N.S.) **5**, 113–116 (1941). [MF 6128]

This note, as the title indicates, contains some completions and additions to a previous paper by the same author [J. Indian Math. Soc. (N.S.) **3**, 118–125 (1938)].

T. Fort (Bethlehem, Pa.).

Minakshisundaram, S. A note on the theory of infinite series. Math. Student **9**, 78–81 (1941). [MF 6004]

The author proposes the following question: "Under what circumstances is $\lim na_n=0$ a necessary condition for the convergence of $\sum a_n$?" He also undertakes to generalize the condition $\lim na_n=0$. Five theorems and several lemmas are proved and a number of corollaries are stated. *T. Fort.*

Martin, Margaret. A sequence of limit tests for the convergence of series. Bull. Amer. Math. Soc. **47**, 452–457 (1941). [MF 4537]

The author proves two similar theorems concerning the convergence of series of positive terms. In the statement of these theorems she utilizes

$$R_n=\frac{u_{n+2}}{u_{n+1}}\div\frac{u_{n+1}}{u_n}.$$

T. Fort (Bethlehem, Pa.).

Oguiewetzki, I. Une généralisation d'un théorème de M. J. Hadamard sur la convergence des séries. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 206–207 (1941). [MF 4829]

Hadamard proved [Acta Math. **27**, 177 (1903)]: If whenever $\sum a_r$ converges $\sum a_rb_r$ converges then $\sum|\Delta b_r|$ converges. In the present note this theorem is extended to series of functions. *T. Fort* (Bethlehem, Pa.).

Sidon, S. Über das Abelsche Summationsverfahren. Studia Math. **9**, 106–108 (1940). (German. Ukrainian summary) [MF 5259]

In order that a sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$ of complex constants be such that a constant C exists for which $\sum\alpha_ku_k$ is summable by Abel's method and

$$\left|\lim_{x\to1}\sum_{k=0}^{\infty}\alpha_ku_kx^k\right|\leqq C\ \text{l.u.b.}_{0<x<1}\left|\sum_{k=0}^{\infty}u_kx^k\right|$$

whenever $\sum u_k$ is summable by Abel's method, it is necessary and sufficient that the sequence α_k be the moment sequence of a function $p(t)$ having bounded variation over $0\leqq t\leqq1$. Proof of sufficiency is straightforward; proof of necessity uses the F. Riesz representation of the most general bounded linear functional on the space of continuous functions. *R. P. Agnew* (Ithaca, N. Y.).

Macphail, M. S. Cesàro summability of a class of series. Bull. Amer. Math. Soc. **47**, 483–487 (1941). [MF 4542]

Let $a(x)$ be a polynomial in x of degree $k-1$, k being a fixed positive integer. Let f_n be a divergent periodic sequence with mean value 0. Then the series $\sum f_na(n)$ is summable C_k but not C_{k-1}. If z is a complex number for which $|z|=1$, $z\neq1$, then $\sum a(n)z^n$ is summable C_k but not C_{k-1}. Formulas for the C_k values of the series are given. *R. P. Agnew.*

Ward, A. J. A remark on Kloosterman's paper "On the convergence of series summable (C, r)." J. London Math. Soc. **16**, 81–82 (1941). [MF 5122]

The author shows that the expression $L_k{}^r(n)\equiv\Delta_k{}^rS_n^{(r)}-h^rS_n^{(0)}$ [the notation is that used in Kloosterman's paper, J. London Math. Soc. **15**, 91–96 (1940); cf. these Rev. **2**, 89] is a linear function of $a_{n+1}, a_{n+2}, \cdots, a_{n+rk}$ with the sum of the coefficients, all of which are positive, being $\frac{1}{2}rh^r(k+1)$. By the use of this expression, the proof of Kloosterman's Theorem 2 can be shortened. *D. Moskovitz.*

Bosanquet, L. S. and Chow, H. C. Some analogues of a theorem of Andersen. J. London Math. Soc. **16**, 42–48 (1941). [MF 4954]

The two series

$$\sum u_n\equiv\sum\binom{\rho+n}{n}(b_{n+1}-b_n),\quad \sum v_n\equiv\sum\binom{\rho+n-1}{n}b_n,$$

in which ρ is a real constant, are related by a formal Abel partial summation. The terms of $\sum u_n$ determine those of $\sum v_n$; the terms of $\sum v_n$ determine those of $\sum u_n$ except for a constant which may be added to each b_n. Let $|C,\alpha|$ denote absolute Cesàro summability of order α. The following theorem is analogous to a theorem of Andersen involving (C,α). (i) If $\rho\neq0$, $\alpha>-1$, and $\sum u_n$ is summable $|C,\alpha|$, then $\sum v_n$ is summable $|C,\alpha+1|$. (ii) If $\rho>0$, $\alpha>-1$, and $\sum v_n$ is summable $|C,\alpha+1|$, there is a unique determination of $\sum u_n$ which is summable $|C,\alpha|$; if $\rho<0$, each determination of $\sum u_n$ is summable $|C,\alpha|$. Analogous results are given for the series $\sum n^{\rho-1}b_n$ and $\sum n^{\rho}(b_n-b_{n+1})$.

R. P. Agnew (Ithaca, N. Y.).

Forsythe, G. E. Riesz summability methods of order r, for $\Re(r)<0$. Duke Math. J. **8**, 346–349 (1941). [MF 4598]

Given a series $u_0+u_1+\cdots$ of complex terms and a complex parameter r, let α_n be defined for $n=1, 2, \cdots$ and $\beta(t)$ for $t\geqq1$ by

$$\alpha_n=\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)^ru_k,\quad \beta(t)=\sum_{k=0}^{[t]-1}\left(1-\frac{k}{t}\right)^ru_k.$$

Let $\sum u_n$ be called summable A_r to L if $\alpha_n\to L$ as $n\to\infty$, and summable B_r to L if $\beta(t)\to L$ as $t\to\infty$. The problem of determining the values of r for which A_r and B_r are equivalent has been partially solved. This paper extends known results by showing that if $r=-1+ih$, where h is real, then A_r and B_r are equivalent if and only if $h=0$.

R. P. Agnew (Ithaca, N. Y.).

Good, I. J. Note on the summation of a classical divergent series. J. London Math. Soc. **16**, 180–182 (1941). [MF 6017]

A series $\sum a_n$ is summable (B,α) to L if $F(x)\to L$ as $x\to\infty$, where

$$F(x)=\int_0^x\alpha^{-1}t^{\rho}e^{-t}\sum_{n=0}^{\infty}[a_nt^n/\Gamma(1+\alpha n)]dt,$$

$p=\alpha^{-1}-1$, $q=t^{1/\alpha}$; $(B, 1)$ is the Borel integral method of summability. Hardy has stated that the series $\sum(-1)^n n!$ is summable (B, α) when $\alpha>1$. This note shows that $\sum(-1)^n n!$ is summable (B, α) only when $1<\alpha\leqq 3$; when $\alpha>3$, $F(x)$ exists but $\lim F(x)$ does not. *R. P. Agnew.*

San Juan, Ricardo. An algorithm for summation of divergent series. Revista Union Mat. Argentina **7**, 71–73 (1941). (Spanish) [MF 5532]

The method of Borel for the summation of power series is generalized by introducing a parameter α, $0<\alpha<1$, and defining the B_α-sum of $\sum a_n z^n$ as

$$\lim_{\alpha\to 1-}\int_0^{+\infty} e^{-t}\left(\sum_{n=0}^{\infty}\frac{a_n}{n!}[t^\alpha z]^n\right)dt.$$

It is shown that analytic continuation is effected throughout the Mittag-Leffler star. *H. S. Wall* (Evanston, Ill.).

Kharchiladze, Philippe. Sur la méthode de sommation de S. Bernstein et W. Rogosinsky. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 697–700 (1941). [MF 4474]

The author gives a few simple consequences of the following theorem. If the series $a_0+a_1+\cdots$ is summable (ρ, α) $(\alpha=1, 2)$ to sum s, then

$$\lim_{n\to\infty}\sum_{k=0}^{n} a_k(\cos k\pi/(2n+1))^\alpha\to s.$$

This theorem (even for every $\alpha=1, 2, 3, \cdots$) is contained in a result of Rogosinski [Math. Z. **25**, 132–149 (1926), especially p. 134]. *A. Zygmund* (Ann Arbor, Mich.).

Garabedian, H. L. Hausdorff methods of summation which include all of the Cesàro methods. Bull. Amer. Math. Soc. **48**, 124–127 (1942). [MF 6191]

Garabedian, Hille and Wall [Duke Math. J. **8**, 193–213 (1941); these Rev. **2**, 278] have characterized the mass functions $\chi(t)$ for which the Hausdorff method $H(\chi)$ includes the Cesàro method $C(n)$, n being a fixed positive integer. This note gives simple examples of mass functions $\chi(t)$ for which the conditions are satisfied when $n=1, 2, 3, \cdots$ and for which accordingly $H(\chi)\supset C(r)$ when $r>-1$. Hausdorff has given examples of mass functions for which $H(\chi)\supset C(r)$ when $r>-1$; in these examples the moment sequences are simple but the mass functions are not.

R. P. Agnew (Ithaca, N. Y.).

Agnew, Ralph Palmer. On methods of summability and mass functions determined by hypergeometric coefficients. Amer. J. Math. **63**, 705–708 (1941). [MF 5618]

Let $\{\lambda_n(\alpha, \beta, \gamma)\}$ be the sequence of coefficients in the power series expansion of the hypergeometric function $F(\alpha, \beta, \gamma; z)$. The sequence $\{\lambda_n(\alpha, \beta, \gamma)\}$ determines a transformation $A(\lambda_n(\alpha, \beta, \gamma))$, in the domain of a general class of methods of summation first considered by Hurwitz and Silverman and by Hausdorff, which defines the hypergeometric method of summation denoted by $(H, \alpha, \beta, \gamma)$ [H. L. Garabedian and H. S. Wall, Trans. Amer. Math. Soc. **48**, 185–207 (1940); cf. these Rev. **2**, 90]. The author derives an equivalence relationship between the hypergeometric and Cesàro methods of summation: $(H, \alpha, \beta, \gamma)\approx C_{\gamma-\alpha-\beta+1}$, $R(\alpha, \beta, \gamma)>0$; $R(\gamma-\alpha-\beta+2)>0$. Interpretations of various restrictions on the parameters are given.

H. L. Garabedian (Evanston, Ill.).

Kuttner, B. The generalized limit of a function. Proc. London Math. Soc. (2) **47**, 142–173 (1941). [MF 4705]

Let $f(x)$ be a real non-negative function, defined for $x>0$ and integrable over each finite interval $0\leqq x\leqq x_1$. Let

$$f_1(x)=\int_0^x f(t)dt$$

and let $f(x)$ be such that $f(x)=o[f_1(x)]$ as $x\to\infty$. A function $A(x)$, bounded and measurable over each finite interval $0\leqq x\leqq x_1$, is called summable (f) to L if the transform $A_f(x)$ defined by

$$A_f(x)=[f_1(x)]^{-1}\int_0^x A(t)f(x-t)dt$$

converges to L as $x\to\infty$. This is the function analogue of the Nörlund method of summability of sequences. Criteria for inclusion among Nörlund methods were obtained by M. Riesz. The problem of this paper is to obtain analogous criteria for methods of the form (f). The problem is solved for functions $f(x)$ having the form

$$f(x)=kx^\lambda+c_1x^{\lambda+1}+c_2x^{\lambda+2}+c_3x^{\lambda+3}+c_4x^{\lambda+4}+r(x),$$

where k, c_1, $\cdots$, c_4, λ are constants for which $k\neq 0$ and $\lambda>-1$ and where n is the least integer not less than λ, $r(x)$ is in $0\leqq x<\infty$ an $(n+4)$th indefinite integral of a function which is $O(e^{\epsilon x})$ as $x\to\infty$ for each $\epsilon>0$. Applications are given to Cesàro and Riesz summability of functions. It is stated in §3.6 that the proof (13 pages) of Theorem 2 is complicated by the fact that here we have no general theorems analogous to the Silverman-Toeplitz-Schur theorems on matrix transformations. Such general theorems are included in a treatment of summability of functions by the reviewer [Bull. Amer. Math. Soc. **45**, 689–730 (1939); cf. these Rev. **1**, 50]. *R. P. Agnew.*

Fort, Tomlinson. Summability and the definition of a limit. Amer. Math. Monthly **49**, 37–44 (1942). [MF 6144]

In analogy to the transform

$$\psi(x)=\int_0^\infty K(x, t)f(t)dt,$$

certain transforms are considered, where the integral is replaced by a "sum" as the inverse of a difference.

O. Szász (Cincinnati, Ohio).

Fichera, Gaetano. Generalizzazione del teorema d'Abel sulle serie di potenze. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 810–820 (1941). [MF 5755]

If the sum of $m>0$ ascending power series $\sum_{i=0}^{\infty}a_i^{(\nu)}\rho^{\mu+\nu i}$ and $n>0$ descending series $\sum b_i^{(\nu)}\rho^{-\mu+\nu i}$, where the a's and b's are real or complex, ρ is positive and the μ's are real and distinct and likewise the ν's, converges for $m+n$ distinct values of ρ, then the separate series converge and the ascending series have a common interval of convergence $\rho\leqq R$ and the descending series a common interval $\rho\geqq r$. If the points of convergence of the original series are arranged in increasing order of magnitude, then the statement holds for r equal to the mth member of this sequence and R equal to the $(m+1)$st member. The author states the theorem to include the important special case where two or more of the μ's or ν's are equal, but the theorem would be false under such conditions, as can be seen by considering the sum of a divergent series and its negative.

P. W. Ketchum (Urbana, Ill.).

Scott, W. T. and Wall, H. S. Value regions for continued fractions. Bull. Amer. Math. Soc. **47**, 580–585 (1941). [MF 5049]

The authors prove that, if $a_2, a_3, a_4, \cdots$ are complex numbers which lie within or upon the parabola $|z|-\Re(z)=\frac{1}{2}$, all the approximants of the continued fraction $1/1+a_2/1+a_3/1+\cdots$ lie within or upon the circle $|z-1|=1$. Further they show that this is a "best" result. The method of proof is partly analytic and partly geometric. Under the above conditions on the numbers a_n the authors obtain also a circular value region for the "even" approximants of the given continued fraction. *W. Leighton.*

Oppenheim, Alexander. Rational approximations to irrationals. Bull. Amer. Math. Soc. **47**, 602–604 (1941). [MF 5052]

The author proves that, if p and q are relatively prime, $q>0$, and if (1) $|x-p/q|<1/q^2$, then p/q is one of the three irreducible fractions p'/q', $(p'+p'')/(q'+q'')$, $(p'-p'')/(q'-q'')$, where p''/q'', p'/q' are two consecutive convergents in the regular continued fraction expansion of the irrational number x; further, one at least of the two fractions $(p'+\epsilon p'')/(q'+\epsilon q'')$, where $\epsilon=\pm 1$, satisfies (1). The editor calls attention to a paper by R. M. Robinson [Duke Math. J. **7**, 354–359 (1940); cf. these Rev. **2**, 149] and to a result of P. Fatou [C. R. Acad. Sci. Paris **139**, 1019–1021 (1904)]. *W. Leighton* (Houston, Tex.).

Krishnaswami Ayyangar, A. A. The role of unit partial quotients in some continued fractions. Math. Student **8**, 159–166 (1940). [MF 5003]

The author studies arithmetical continued fraction representations of certain quadratic surds. A typical theorem is the following: In the development of $R^{\frac{1}{2}}/Q$ as a continued fraction to the nearest integer there cannot occur more than one complete quotient of the form $(p+q+(p^2+q^2)^{\frac{1}{2}})/p$ in the recurring period, while in the singular continued fraction development such a complete quotient can never occur. *W. Leighton* (Houston, Tex.).

Integral Equations

Doubrowsky, W. Équations intégrales du type de Fredholm dont le noyau est une fonction d'élément et d'ensemble dans un espace abstrait. Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 403–420 (1941). (Russian. French summary) [MF 4563]

Let A be an abstract space, $\mathfrak{M}$ a class of sets of A, containing all elements x of A, differences and denumerable sums of sets of $\mathfrak{M}$, and the space A itself. Let $k(x, e)$ be a function depending on $x \in A$, $e \in \mathfrak{M}$. It is assumed that $k(x, e)$ is completely additive in e for x fixed, and is measurable relative to $\mathfrak{M}$ for a fixed e. It is also assumed that $k(x, e)$ satisfies the following conditions. (I) Let $\bar{k}(e) = \sup \sum_{i=1}^{n} \sup |k(x, e_i)|$, where the inner sup is taken over all x for a fixed decomposition of e into a sum of disjoint sets $e_i \in \mathfrak{M}$, and the outer sup is taken over all such decompositions. Then $\bar{k}(A)<\infty$. (II) There exists a completely additive function $k_1(e)$ such that for any $\epsilon>0$ it is possible to decompose A into a sum of disjoint $e_i \in \mathfrak{M}$ such that for any x_i, $x_i' \in e_i$ and any e, $|k(x_i, e)-k(x_i', e)|<\epsilon k_1(e)$, $i=1, 2, \cdots$. The author considers integral equations

$$\phi(x)=f(x)+\lambda\int_{A_y} k(x, dA_y)\phi(y),$$

$$\psi(e)=g(e)+\lambda\int_{A_y} k(y, e)\psi(dA_y),$$

where integrals are taken in the sense of Kolmogoroff, $f(x)$ is measurable, $g(e)$ is completely additive, and shows that the classical Fredholm theory can be extended to this more general case. At the end of the paper the author discusses the case where (II) is replaced by a less restrictive condition, and also considers various examples.

J. D. Tamarkin (Providence, R. I.).

Kupradze, V. D. On the theory of integral equations with the principal Cauchy value integrals. Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 23–28, 227–232 (1941). (Russian) [MF 5294]

The author introduces the following notation: γ is a simple closed curve of finite curvature; s, t are complex coordinates of points on γ; known functions $a(s)$, $c(s)$, $K(s, t)$, $B(s, t)$ and the unknown function $\varphi(s)$ satisfy Hölder's condition. The integral operator under consideration is

$$K(\varphi(s))=a(s)\varphi(s)-\lambda c(s)\int_\gamma [(t-s)^{-1}K(s, t)+B(s, t)]\varphi(t)dt,$$

where the integral is in the sense of principal Cauchy value; $\bar{K}(\bar{\varphi})$ is the operator in whose kernel s and t are interchanged. Simple proofs of the following known results are given. In order that $K(\varphi(s))=f(s)$ should have a solution it is necessary and sufficient that $f(s)$ be orthogonal to all the solutions of $\bar{K}(\bar{\varphi})=0$; the number of distinct solutions of $K(\varphi)=0$ minus the number of distinct solutions of $\bar{K}(\bar{\varphi})=0$ is $(\pi i)^{-1}\int_\gamma d \log q(s)$, where

$$q(s)=\frac{a(s)+\lambda\pi i b(s)}{a(s)-\lambda\pi i b(s)}.$$

W. J. Trjitzinsky (Urbana, Ill.).

Nazarov, N. Non-linear integral equations of Hammerstein's type. Acta [Trudi] Univ. Asiae Mediae. Ser. V-a., fasc. **33**, 79 pp. (1941). (Russian. English summary) [MF 5509]

The author studies equations

$$(1)\qquad \varphi(x)=\lambda\int_0^1 K(x, y)f(y, \varphi(y))dy,$$

$$(2)\qquad \varphi(x)=\lambda\int_0^1 \Gamma(x, y, \varphi(y))dy.$$

Some of the results are as follows: (A) (2), with $\lambda=-1$ and $\Gamma(x, y, z)=\int_0^1 K(x, t)h(t, y, z)dt$, has a solution, provided $K(x, y)$ is symmetric of square integrable over $0\leqq x, y\leqq 1$, while (a) $h \subset$ Lip. (in z; sufficiently small Lip. constant) and (b) $h^2<A+B^2z^2$ (small B). (B) If $\Gamma=$c.p.s. (convergent power series) in z $(0\leqq x, y\leqq 1; |z|<R)$, then (2) has a solution $=$c.p.s. in λ ($|\lambda| \leqq$sufficiently small r). (C) If $\Gamma \subset$ Lip. (in z) for $\{0\leqq x, y\leqq 1, |z|\leqq M\}$, then (2) has a solution for $|\lambda|$ sufficiently small (obtainable by successive approximations). (D) Let $\varphi_0(x)$ be a solution of (1) for $\lambda=\lambda_0$, $f(y, \varphi_0(y)+z)=A_0(y)+A_1(y)z+\cdots$ (convergent in a z-domain), $K(x, y)$ be definite symmetric, $A_1(y)\neq 0$ on $(0, 1)$ and let $\varphi_1(y)$ be the characteristic function corresponding

to λ_0 (if λ_0 is a characteristic number). Then: (a) (1) has a unique solution, analytic at λ_0 and tending to $\varphi_0(x)$, as $\lambda \to \lambda_0$, provided λ_0 is not a characteristic number of $K(x, y)\partial f(y, \varphi_0(y))/\partial \varphi_0(y)$. (b) There are two solutions, as in (a), whenever λ_0 is a characteristic number and $\int_0^1 A_0\varphi_1 dy = 0$, $A_0 \not\equiv 0$. (c) (1) has a solution = c.p.s. in $(\lambda-\lambda_0)^{1/(n-1)}$ and tending to $\varphi_0(x)$, as $\lambda \to \lambda_0$, provided $A_0 = A_2 = \cdots = A_{n-1} \equiv 0$, $A_n \not\equiv 0$ (λ_0 being a characteristic number). (d) (1) has a solution = c.p.s. in $(\lambda-\lambda_0)^{1/n}$ ($\to \varphi_0(x)$, as $\lambda \to \lambda_0$), whenever λ_0 is a characteristic number, while $\int_0^1 A_0\varphi_1 dy \neq 0$, $A_2 = A_3 = \cdots = A_{n-1} \equiv 0$, $A_n \not\equiv 0$. Some results analogous to those in Theorem (D) are obtained for (2). The methods used by the author are non-topological. His total contribution to the field of integral equations is substantial. In view of the vastness of the subject many aspects of this field are still left open (the theory of singular solutions, for instance). *W. J. Trjitzinsky.*

Gunther, N. Sur les équations intégrales de troisième espèce. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 683–686 (1941). [MF 4470]

The theory of the integral equation

$$\phi(x) = \lambda \int_a^b G(x, y)k(y)\phi(y)dy + f(x), \tag{1}$$

where $k(y)$ is a bounded function changing its sign in the interval (a, b) and where the symmetric kernel $G(x, y)$ is continuous and positive definite, is approached in a way different from the well-known methods of Hilbert, Marty and Garbe. Using the characteristic values and functions of $G(x, y)$ and Mercer's theorem, the author forms a kernel $G_1(x, y)$ which is the symbolic square root of $G(x, y)$, such that

$$G(x, y) = \int_a^b G_1(x, z)G_1(z, y)dz.$$

This enables him to deduce all properties of the integral equation (1) from the theory of the symmetric kernel $G_1(x, y)$. It may be noted that this procedure is essentially the method by which Hilbert [Nachr. Ges. Wiss. Göttingen 1906, 209 ff.; Grundzüge der allgemeine Theorie der linearen Integralgleichungen, 1912, pp. 156 ff.] treated the theory of quadratic forms in infinitely many variables, which is equivalent to the theory of the integral equations of third kind [cf. Enzykl. Math. Wiss., II C 13, no. 41, part 4, pp. 1567 ff.]. *E. Hellinger* (Evanston, Ill.).

Reissner, Eric. On a class of singular integral equations. J. Math. Phys. Mass. Inst. Tech. **20**, 219–223 (1941). [MF 5065]

The Wiener-Hopf method [Paley and Wiener, Trans. Amer. Math. Soc. **35**, 761–791 (1933), section 4] for the case $h(x) = 0$ is shown to hold for the nonhomogeneous integral equation $f(x) = h(x) + \int_0^\infty k(x-y)f(y)dy$, $0 < x < \infty$. The conditions imposed are: $k(x)e^{\lambda|x|} \in L_2(-\infty, \infty)$ for $\lambda < 1$ and $h(x) = 0$ for $x < 0$, and $O(e^{ax})$, $a < 1$, for $x > 0$.

D. G. Bourgin (Urbana, Ill.).

Feller, Willy. On the integral equation of renewal theory. Ann. Math. Statistics **12**, 243–267 (1941). [MF 5411]

This paper is devoted to the study of the behavior of the solutions of the integral equation

$$u(t) = g(t) + \int_0^t u(t-x)f(x)dx \tag{*}$$

and certain generalizations of it; $f(t)$ and $g(t)$ are given non-negative functions. Under various forms (*) appears in such topics as (a) population theory, (b) the theory of industrial displacements and (c) the genera[illegible]heory of self-renewing aggregates. While the results of Pa[illegible]d Wiener [Trans. Amer. Math. Soc. **35**, 785–791 (193[illegible]uld be used to deduce the existence of the solution (*), [illegible] non-negative character of $f(t)$ and $g(t)$ makes the problem much too simple to justify such refined tools. Using the properties of completely monotone functions, the author deduces the existence and also some properties of the solution of (*). Asymptotic properties of the integral mean value of $u(x)$ are studied with the aid of Tauberian theorems for Laplace integrals. The nature of the expansions of $u(t)$ is discussed and a critical analysis of some of the past work of Lotka and Hadwiger follows. A bibliography covering papers on the integral equations of renewal theory from 1939 to date is appended. *A. E. Heins* (Lafayette, Ind.).

Lodi, Maria. Risoluzione di una particolare equazione di Volterra in due variabili. Boll. Un. Mat. Ital. (2) **3**, 391–393 (1941). [MF 5708]

Si espone una semplice soluzione della seguente equazione integrale di Volterra in due variabili:

$$\phi(x, y) = f(x, y) + \int_0^y dx_1 \int_0^y e^{\alpha(x-x_1)+\beta(y-y_1)}\phi(x_1, y_1)dy_1.$$

Author's summary.

Muschelišvili, N. I. Über die Lösung der fundamentalen Randwertaufgaben des Newtonschen Potentials. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, no. 4, 3–26 (1940). (Russian. German summary) [MF 4645]

Let D be a domain bounded from outside by a surface S_0 and from inside by m surfaces $S_1, S_2, \cdots, S_m$. Let $S = S_0 + S_1 + \cdots + S_m$. The classical method of solving the Dirichlet problem for D reduces it to an integral equation

$$\mu(p) + \int_S K(p, q)\mu(q)dS_q = f(p). \tag{*}$$

A disadvantage of this method consists in the fact that the homogeneous equation (*) has m solutions, which introduces various complications, at least from a practical point of view. The author considers a modified Dirichlet problem: to find a function W harmonic in D which is representable as a potential of double layer on S, and which satisfies conditions $W(p) = f(p) + a_j$ on S_j, $j = 0, 1, \cdots, m$, where $a_0 = 0$, a_j, $j = 1, 2, \cdots, m$, are constants to be determined later. Instead of the integral equation of type (*) to which this problem naturally reduces the author considers a modified integral equation

$$\mu(p) + \int_S K(p, q)\mu(q)dS_q + \int_S k(p, q)\mu(q)dS_q = f(p), \tag{**}$$

where the supplementary kernel $k(p, q)$ is an arbitrary continuous function restricted only by the conditions that $k(p, q) = \rho_j(q)$ when p and q are on S_j, $j = 1, 2, \cdots, m$, and $k(p, q) = 0$ otherwise. Under the assumption $\int_{S_j} \rho_j dS \neq 0$ it is shown that the homogeneous equation (**) has no solution distinct from zero and that, for a suitable choice of constants a_j, the solution of (**) will be also a solution of (*). Introducing a parameter λ in front of integral terms in

(**) the author shows that, if

$$0<\int_{S_j}\rho_j dS<2,\quad j=1,2,\cdots,m;\quad -2<\int_{S_0}\rho_0 dS<0,$$

then all characteristic values are real and outside of the closed interval $[-1, 1]$. Analogous methods can be applied to solve Neumann's problem and electrostatical problem.
J. D. Tamarkin (Providence, R. I.).

Caligo, Domenico. Un criterio sufficiente di stabilità per le soluzioni dei sistemi di equazioni integrali lineari e sue applicazioni ai sistemi di equazioni differenziali lineari. Atti del secondo Congresso dell'Unione Matematica Italiana, Bologna, 4–6 Aprile 1940, pp. 177–185. [MF 5761]

The methods of a preceding paper of the author [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 497–506 (1940); these Rev. **3**, 48] are herein continued and extended to the case of a system of first order ordinary linear differential equations. *W. T. Reid* (Chicago, Ill.).

Birkhoff, George David. Rectilinear drawing. Rice Inst. Pamphlet **28**, 51–76 (1941). (4 plates) [MF 6079]

A simplified form of the author's paper: "On drawings composed of uniform straight lines" [J. Math. Pures Appl. (9) **19**, 221–236 (1940); see these Rev. **2**, 313].
T. H. Hildebrandt (Ann Arbor, Mich.).

Nardini, Renato. Studio e risoluzione di un'equazione funzionale del tipo misto. Ann. Scuola Norm. Super. Pisa (2) **9**, 201–213 (1940). [MF 5462]

This paper treats a functional equation of the form

$$\psi(x,y)=f(x,y)+\lambda\int_0^b K(x,y,z)\psi(x,z)dz+A[x,y,\psi(\overset{x}{0},\overset{b}{0})],$$

where $f(x, y)$ is a given continuous function on $0\leqq x\leqq 1$, $0\leqq y\leqq b\leqq 1$, $K(x, y, z)$ is continuous on $0\leqq x\leqq 1$, $0\leqq y, z\leqq b$, and

$$A[x,y,\psi(\overset{x}{0},\overset{b}{0})]$$

is a real-valued functional of the point arguments x, y and the functional values of $\psi(t, z)$ on the rectangle $0\leqq t\leqq x$, $0\leqq z\leqq b$. It is assumed throughout that for $0\leqq x\leqq 1$ the value λ is not a characteristic value of the kernel $K(x, y, z)$. Under certain hypotheses of boundedness and continuity of the functional A, which, due to their complicacy, will not be specifically stated here, the author proves the existence of a value $l>0$ such that there exists a continuous solution of this equation on $0\leqq x\leqq l$, $0\leqq y\leqq b$. The uniqueness of solution is established in case the functional A satisfies a certain type of Lipschitz condition. Finally, conditions are given which assure the existence and uniqueness of the solution on $0\leqq x\leqq 1$, $0\leqq y\leqq b$. *W. T. Reid.*

NUMERICAL AND GRAPHICAL METHODS

***Southwell, R. V. Relaxation Methods in Engineering Science. A treatise on approximate computation.** Oxford Engineering Science Series, Oxford University Press, New York, 1940. vii+252 pp. $5.00.

The term "relaxation" is applied by the author to a method of solution of equations of statics by successive approximations. The method is first applied to pin-jointed frameworks, by first preventing motion of all the joints by proper constraints, then releasing one joint at a time by allowing it to deflect in a certain direction while the load applied at that joint is transmitted to the adjacent joints which are meanwhile held fast; by releasing the various joints in different directions over and over again, one approaches the displacements and reactions of the frame which constitute the solution of the problem in question. These relaxation methods are also applied to continuous beams, to plane frameworks having rigid joints, to electrical networks and to other linear problems. The calculation is speeded up under certain conditions by relaxing whole "blocks" or "groups" of the framework members. Further applications of the method are given to vibrating systems in connection with the calculation of natural modes and frequencies of vibrations. Convergence proofs are given.

A great deal of the material of this book has appeared as magazine articles for several years by the author and various collaborators. The method of relaxation is closely related to similar methods advocated in this country by Hardy-Cross. Brief mention is also made of continuous problems, for instance the solution of Laplace's equation in two dimensions by means of an approximating network and replacing the differential equation by a difference equation; problems of elastic stability (buckling), as well as nonlinear systems, are treated briefly, but they are not treated in sufficient detail to save the reader from looking up the references. The book is attractively written and printed and will be welcomed by workers in this field.
H. Poritsky (Schenectady, N. Y.).

***Emde, Fritz. Tafeln elementarer Funktionen.** B. G. Teubner, Leipzig, 1941. xii+181 pp. R.M. 12. (83 figures; text in German and English)

Table of contents: Powers. Factorials. Tables to facilitate computation with complex numbers. Quadratic equations. Cubic equations. Equations of fourth degree. Angles in degrees. Angles in right angles. Angles in radians. Special functions (e^{-x^2}, Planck's function, the fundamental solution of the heat equation, Langevin's function $\coth x-x^{-1}$, $\sinh x\sin x$, $\sinh x\cos x$, $\cosh x\sin x$, $\cosh x\cos x$, $\rho\tan\rho$, $\rho\cot\rho$, roots of $\rho\tan\rho=\text{const.}$, $\rho^{-1}\tan\rho=\text{const.}$). Transcendental equations. Circular and hyperbolic functions of a complex variable. The function $\tan(ri^{\frac{1}{2}})/(ri^{\frac{1}{2}})$. Tschebyscheff's polynomials. Approximations by polynomials. Remarks on numerical computation.

***Table of Natural Logarithms. Vol. III.** Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 65-2-97-33; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1941. xviii+501 pp. $2.00.

For the first two volumes, see these Rev. **2**, 366. The present volume contains the logarithms of the decimal numbers from .0001 to 5.0000 to sixteen decimal places.

Heuman, Carl. Corrections in "Tables of complete elliptic integrals." J. Math. Phys. Mass. Inst. Tech. **20**, 336 (1941). [MF 5071]

Cf. these Rev. **2**, 239.

Thompson, Catherine M. Tables of percentage points of the incomplete beta-function. Prefatory note by E. S. Pearson; description of the calculation by L. J. Comrie and H. O. Hartley; methods of interpolation by H. O. Hartley. Biometrika **32**, 151–181 (1941). [MF 5742]

The tables gives the values of x satisfying the equation

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_0^x x^{p-1}(1-x)^{q-1}dx=P$$

for $P=0.005$, 0.01, 0.025, 0.05, 0.10, 0.25 and 0.50, for $2q=1$ to 10 at unit intervals, also $2q=12$, 15, 20, 24, 30, 40, 60, 120, ∞, and for $2p=1$ to 30 at unit intervals, also $2p=40$, 60, 120 and ∞. The entries for $2q$ and $2p$ beyond 30 are chosen so that their reciprocals are in arithmetic progression. The entries for x are given always to five significant figures, which means, in the worst case, ten places of decimals. The introduction explains the purposes of the table, the methods of calculation and appropriate methods of interpolation. *W. E. Milne* (Corvallis, Ore.).

Comrie, L. J. and Hartley, H. O. Table of Lagrangian coefficients for harmonic interpolation in certain tables of percentage points. Biometrika **32**, 183–186 (1941). [MF 5743]

This table gives to six places of decimals the Lagrangian coefficients for 7-term interpolation formulas adapted for use with the tables of the preceding review. Ordinary polynomial interpolation is provided in the range 10–20 and harmonic interpolation in the range 16–120. Entries are given for all integral values of the argument in the range 10–120 not tabulated in the original table. *W. E. Milne.*

Beckerley, James G. The calculation of arg $\Gamma(ia+1)$. Indian J. Phys. **15**, 229–232 (1941). [MF 5712]

A modification of the known series for the argument of the complex gamma function $\Gamma(ia+1)$, which is more suited to numerical computation, is given in the form

$$\arg\Gamma(ia+1)=a(S_1^N-C)-\sum_{n=1}^{N}\arctan\frac{a}{n}-\sum_{n=1}^{P}\frac{(-1)^n\Delta S_{2n+1}^N}{2n+1}a^{2n+1}+R_P,$$

where

$$S_\mu^\nu=\sum_{n=1}^{\nu}\left(\frac{1}{n}\right)^\mu,\quad \Delta S_\mu^\nu=S_\mu^\infty-S_\mu^\nu,\quad |R_P|\leqq\frac{\Delta S_{2P+3}^N}{2P+3}a^{2P+3}.$$

The paper includes short tables of values of ΔS_{2n+1}^N ($1\leqq n, N\leqq 5$) and of $\arg\Gamma(ia+1)$, $0\leqq a\leqq 2$. *M. C. Gray.*

Nomura, Yûkiti. The electrostatic problems of two equal parallel circular plates. Proc. Phys.-Math. Soc. Japan (3) **23**, 168–180 (1941). [MF 4519]

Calculations are made with the aid of tables to seven decimal places of the integral

$$g(m,n,x)=\int_0^\infty e^{-xt}J_m(t)J_n(t)dt/t$$

for half integral values of m and n such that $0<m+n\leqq 8$. The values of x lie mostly in the range $0.4\leqq x\leqq 5$, but the values 10 and 20 are used for $m+n\leqq 6$ and $m+n\leqq 4$, respectively. Tables are also given for the coefficients in certain expansions of type $\sum C_m^k G_{m,k}^n$, where

$$G_{m,k}^n=(2m+4k+1)g(2n+\tfrac{1}{2}, m+2k+\tfrac{1}{2}, x).$$

The results are applied to the problem of the condenser and also to the general problem of discs charged to given potentials. A calculation is made of the force between two conducting plates. As the distance between the plates increases from zero the force is found to decrease more rapidly for equal and opposite charges than for equal charges.

H. Bateman (Pasadena, Calif.).

Sharp, Henry S. A comparison of methods for evaluating the complex roots of quartic equations. J. Math. Phys. Mass. Inst. Tech. **20**, 243–258 (1941). [MF 5068]

A classification and examination of numerical methods for solving biquadratic equations with real coefficients and four complex (non-real) roots: type I: methods for evaluating absolute value and argument of roots; type II: methods for evaluating the real and pure imaginary parts; type III: methods for determining the two real quadratic factors; type IV: methods depending on special properties of the roots (for example, two pure imaginary roots; real parts of roots numerically equal but opposite in sign; absolute value of roots equal). Besides considering Graeffe's method, particular attention is given to the following more recent papers: W. V. Lyon, J. Math. Phys. Mass. Inst. Tech. **3** (1924), Type I; Y. H. Ku, J. Math. Phys. Mass. Inst. Tech. **5** (1926), Type III; F. L. Hitchcock, J. Math. Phys. Mass. Inst. Tech. **17** (1938), Type IV; E. B. Wilson, Rep. Nat. Adv. Comm. Aeronaut. 7, serial 6917. The author contributes variants to types II, III, IV.

A. J. Kempner (Boulder, Colo.).

Pawley, Myron G. New criteria for accuracy in approximating real roots by the Newton-Raphson method. Nat. Math. Mag. **15**, 111–120 (1940). [MF 5687]

Lin, Shih-Nge. A method of successive approximations of evaluating the real and complex roots of cubic and higher-order equations. J. Math. Phys. Mass. Inst. Tech. **20**, 231–242 (1941). [MF 5067]

If in $f(x)=x^n+a_1x^{n-1}+\cdots+a_n=0$ one pair of roots, real or not real, is small in absolute value compared with the other roots, $\phi(x)=x^2+cx+d$, with $c=a_{n-1}/a_{n-2}$, $d=a_n/a_{n-2}$, will be an approximate factor of $f(x)$. Let the exact result be

$$f(x)=(x^{n-2}+b_2x^{n-3}+\cdots+b_n)(x^2+cx+d)+Px+Q;$$

then a new approximate factorization of $f(x)$ is attempted by using $x^2+c_1x+d_1$, with $c_1=(a_{n-1}-db_{n-1})/b_n$, $d_1=a_n/b_n$, leading to

$$f(x)=(x^{n-2}+b_2'x^{n-3}+\cdots+b_n')(x^2+c_1x+d_1)+P_1x+Q_1.$$

This iterative process is continued. If $P, P_1, \cdots, P_\mu, \cdots\to 0$ and $Q, Q_1, \cdots, Q_\mu, \cdots\to 0$, the quadratic equation $x^2+c_\mu x+d_\mu=0$ will yield an approximation to two roots of $f(x)=0$. An examination of the conditions under which $P_\mu\to 0$, $Q_\mu\to 0$ is not given. As a working rule it is noted that for "convergence" of the process the c_μ must approach a limit and the d_μ must approach a limit, and the process is to be continued until the c_μ, d_μ are not appreciably changed by a few steps of the process. In case the P_μ, Q_μ do not approach 0, the process is "not convergent." Elementary transformations on $f(x)$ may then make the process convergent. Several numerical examples for 4th, 6th and 8th degree equations with real coefficients are given. Solutions are given only to 3 or 4 significant figures in the real and pure imaginary parts. *A. J. Kempner* (Boulder, Colo.).

Plummer, H. C. The numerical solution of a type of equation. Philos. Mag. (7) **32**, 505–512 (1941). [MF 6048]

Finite asymptotic formulas are obtained for the solutions of several equations of the form $\tan x = xf(x)$ and $\cos x \cosh x = \pm 1$. Some of these formulas give all the solutions correct to as many as 4 decimals.

P. W. Ketchum (Urbana, Ill.).

Pipes, Louis A. The solution of a.c. circuit problems. J. Appl. Phys. **12**, 685–691 (1941).

Numerical computations of steady state conditions in electrical networks require, in general, inversion of impedance matrices. As these consist of complex elements, the evaluations become rather tedious. It is shown that, by expressing the real and imaginary components of the inverse in terms of the corresponding components of the original matrix and by applying well-known numerical methods, circuit computations can be carried out with great ease by means of a calculating machine. *H. G. Baerwald.*

Dwyer, P. S. The evaluation of linear forms. Psychometrika **6**, 355–365 (1941). [MF 5892]

This paper is concerned with the numerical evaluation of one or more linear forms (1) $\sum_{j=1}^{n} a_{ij}x_j$, $i = 1, 2, \cdots, p$, where the x_i are defined as solutions of (2) $\sum_{\nu=1}^{n} b_{\mu\nu}x_\nu = c_\mu$, $\mu = 1, \cdots, n$. This evaluation is effected by an application of the "abbreviated Doolittle" method without explicitly solving for the x_ν from (2). Similarly one or more forms (1) are expressed as forms in $y_1, \cdots, y_n$, where the y_i are given numerical linear functions of the x_i. *A. L. Foster.*

Bingham, M. D. A new method for obtaining the inverse matrix. J. Amer. Statist. Assoc. **36**, 530–534 (1941). [MF 5884]

Let A be a non-singular square matrix of order n and $f(\lambda)$ its characteristic polynomial. It is known that $f(A) = 0$. Upon multiplication by A^{-1} we get $(-1)^{n-1}p_nA^{-1} = A^{n-1} - p_1A^{n-2} + \cdots + (-1)^{n-1}p_{n-1}A^0$, where p_r is the rth elementary symmetric function of the characteristic roots. Now the sum of the elements in the main diagonal of A^k is precisely the sum of the kth powers of those roots; hence p_r is readily expressed by means of the principal diagonal elements of A^k, $k = 1, \cdots, r$. The method is illustrated by an example. *W. Feller* (Providence, R. I.).

Tuckerman, L. B. On the mathematically significant figures in the solution of simultaneous linear equations. Ann. Math. Statistics **12**, 307–316 (1941). [MF 5416]

This paper discusses computational errors in the solutions of simultaneous linear equations. It is recommended that each partial result r_i should be written in the form $r_i(1 \pm \epsilon_i)$, where ϵ_i is the accumulated relative uncertainty. Each unknown x_j is then found in the form $x_j(1 \pm \epsilon_j)$. If ϵ_j is intolerably large, the computations should be performed more accurately. In some problems, computational errors must be kept small in comparison with errors of observational origin. *T. E. Sterne* (Aberdeen, Md.).

Hoel, Paul G. On methods of solving normal equations. Ann. Math. Statistics **12**, 354–359 (1941). [MF 5422]

The author shows, by illustrative examples, that three standard routines for computing the inverse of a numerical matrix, namely, those of Aitken, Albert and Doolittle, become essentially identical upon appropriately arranging the work of each. *A. L. Foster* (Berkeley, Calif.).

Vasiļevskis, S. Scheme for the solution of normal equations on the calculating machine. Acta Univ. Latviensis [Latvijas Univ. Raksti] (3) no. **11**, 3–12 (1940) = Publ. Observ. Astr. Univ. Lettonie, no. **4** (1940). (English. Latvian summary) [MF 5150]

This paper describes in detail a particular arrangement, on the computing sheets, of a Gaussian solution of the normal equations in the method of least squares. A multiplying machine is used. A numerical example is shown of the arrangement when there are five unknowns.

T. E. Sterne (Aberdeen, Md.).

Hildebrand, Francis B. The approximate solution of singular integral equations arising in engineering practice. Proc. Amer. Acad. Arts Sci. **74**, 287–295 (1941). [MF 5978]

An extension of Crout's method [J. Math. Phys. Mass. Inst. Tech. **19**, 34–92 (1940); these Rev. **1**, 127] for the approximate solution of integral equations to cases where the solution has singularities of known character at certain points. The unknown function is approximated by a polynomial plus terms involving the singularities with undetermined coefficients. A system of linear equations on the coefficients of the polynomial and of the singularities is obtained by requiring the integral equation to be satisfied at a certain set of points, which need not be equally spaced but may be crowded together near the singularity, or by minimizing the integral of the square of the error. The procedure is tested and solutions accurate to within 1% are found for two physical problems whose exact solutions are known, namely (a) the induced charge on a conducting line segment in a two dimensional field, and (b) the torsion of an elastic bar with triangular (equilateral) cross section.

P. W. Ketchum (Urbana, Ill.).

Hildebrand, F. B. and Crout, P. D. A least square procedure for solving integral equations by polynomial approximation. J. Math. Phys. Mass. Inst. Tech. **20**, 310–335 (1941). [MF 5070]

This is a refinement of a method in a previous paper [J. Math. Phys. Mass. Inst. Tech. **19**, 34–92 (1940); these Rev. **1**, 127]. As in that paper, the unknown function is approximated by a linear combination of standard polynomials with undetermined coefficients. On minimizing the integral of the square of the error made by substitution of this polynomial in one member of the integral equation, there is obtained a system of linear equations on the coefficients which is closely related to the system of the earlier paper (obtained by requiring the error to vanish at a certain number of equally spaced points). Examples are given to show that some problems can be solved by this method which cannot be effectively handled by the previous method. However, both methods are unsatisfactory in case the exact solution fluctuates too rapidly. *P. W. Ketchum.*

Collatz, L. Vergleich der Integralgleichungsmethode von Bucerius mit dem Ritzschen Verfahren zur genäherten Lösung von Differentialgleichungen. Astr. Nachr. **271**, 116–120 (1941). [MF 4610]

The problem $\ddot{x} = f(t, x, \dot{x})$, $x(0) = x(a) = 0$ may be transformed into $x(t) = -\int_0^a G(t, s) f[s, x(s); \dot{x}(s)]ds$; to obtain an approximate solution a finite trigonometric series may be inserted for x. This method is compared with the Ritz method with which it coincides in some cases.

K. Friedrichs (New York, N. Y.).

Busche, Erich. Zur Integration der ballistischen Hauptgleichung. Deutsche Math. 6, 97–99 (1941). [MF 5808]

The author described a numerical procedure to supplant a graphical method proposed by A. Klose [Deutsche Math. 2, 473–479 (1937)]. *W. E. Milne* (Corvallis, Ore.).

Sauer, R. und Pösch, H. Anwendungen des Adamsschen Integrationsverfahrens in der Ballistik. Ing.-Arch. 12, 158–168 (1941). [MF 5791]

The authors describe and illustrate with numerical examples three methods of integrating the fundamental differential equations for the trajectory of a projectile, using quadrature formulas involving differences of the first three orders. Their method is no improvement over that employed by Moulton in 1918 for solving the equations of ballistics. *W. E. Milne* (Corvallis, Ore.).

Mikeladze, Sch. Verallgemeinerung der Methode der numerischen Integration von Differentialgleichungen mit Hilfe der Formeln der mechanischen Quadratur. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 7, 47–63 (1940). (Russian. German summary) [MF 5303]

Given the differential equation

$$y^{(n)}(x)=f(x, y, y', \cdots, y^{(n-1)}), \tag{1}$$

assumed to have a solution $y(x)$ such that $y(x_0)=y_0$, $y^{(i)}(x_0)=y_0^{(i)}$, $i=1, 2, \cdots, n-1$. The author uses the Lagrange interpolation formula and Taylor's formula to obtain an approximate expression for $y^{(k)}_{(a+ah)}$ $(k=0, 1, \cdots, n-1)$, where $a+ah$ is a certain point of the interval $x_0 \leqq x \leqq x_0+\mu h$, in terms of $y(a)$, $y^{(i)}(a)$, $i=1, 2, \cdots, n-1$, with remainder involving $y^{(n)}(x)$ and given in integral form. Assume known the values of $y(x)$, $y'(x)$, $\cdots$, $y^{(n)}(x)$ for $x=x_0, x_0+h, \cdots, a-h, a$. The said formulae, combined with the differential equation (1), yield the approximate values of $y(a+h)$, $y^{(i)}(a+h)$, $i=1, 2, \cdots, n$. In a similar manner we find $y(a+2h)$, $y^{(i)}(a+2h)$, and so on. Formulae are given for halving the interval-difference h. The accuracy of the results thus obtained is discussed in detail for the equation $dy/dx=f(x, y)$, and numerical illustrations are given. *J. A. Shohat* (Philadelphia, Pa.).

Mikeladze, Sch. E. On the question of numerical integration of partial differential equations by means of nets. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 249–254 (1940). (Russian) [MF 5274]

In his book [Numerical Methods for the Solution of Partial Differential Equations, Moscow, 1936] the author investigated the approximate solution of the boundary value problem for

$$a\,\partial^2u/\partial x^2+c\,\partial^2u/\partial y^2+d\,\partial u/\partial x+e\,\partial u/\partial y-gu=f$$

using the method of finite differences. An upper bound for the error which arises in applying his method is given in the paper. The equations $\Delta u(x, y)=\phi(x, y)$ and $\Delta u(x, y, z)=\phi(x, y, z)$ are considered as examples for the application of the method mentioned above. *S. Bergman.*

Grammel, R. Plückersche Koordinaten als Hilfsmittel bei technischen Aufgaben. Ing.-Arch. 12, 169–189 (1941). [MF 5792]

An equation of the form $A(t)u(r)+B(t)v(s)+C(t)=0$ can be solved graphically by interpreting $u(r)$ and $v(s)$ as coordinates of an index line meeting the scale $x(t)=A(t)/C(t)$, $y(t)=B(t)/C(t)$ at a point the mark of which gives the value of t. The method is applied to the integration of a differential equation of one of the following types:

$$A(x)u(y')+B(x)v(y)+C(x)=0,$$
$$A(y)u(y')+B(y)v(x)+C(y)=0,$$
$$A(y')u(x)+B(y')v(y)+C(y')=0.$$

In each of these three cases the slope y' of the integral curve passing through the point x, y can be obtained by the graphical procedure mentioned above. *W. Prager.*

Bickley, W. G. Experiments in approximating to solutions of a partial differential equation. Philos. Mag. (7) 32, 50–66 (1941). [MF 5090]

The differential equation which is treated is the "Fourier equation"

$$\frac{\partial^2\theta}{\partial\xi^2}-\frac{\partial\theta}{\partial\tau}=0. \tag{1}$$

A solution is sought over the interval $0<\xi<1$ which vanishes initially and satisfies the boundary conditions $\theta=1$ at the boundary $\xi=0$ and $\partial\theta/\partial\xi=0$ at $\xi=1$. The exact solution is compared numerically with several approximate solutions. The latter are obtained by assuming for θ for each τ a polynomial in ξ which satisfies the boundary conditions at $\xi=0$, $\xi=1$ and satisfies the differential equation (1) only approximately. By choosing the polynomial so that it has one arbitrary coefficient, and determining the latter so that the error as given by the left-hand member of (1) vanishes at $\xi=1$, a first order differential equation is obtained for this coefficient as a function of τ. The available constant in the solution of this equation is determined so as to satisfy the initial condition for θ at one point. Further approximations are obtained by equating to zero the integral of the error over the whole range of ξ, and by minimizing the integral of the square of the error. Similar approximations involving two arbitrary coefficients are also obtained and compared numerically with the exact solution. A further solution of (1) is also treated in which the boundary condition at $\xi=0$ is replaced by $1-\theta=-\rho(\partial\theta/\partial\xi)$. Again the exact solution is compared with similar approximate solutions. *H. Poritsky* (Schenectady, N. Y.).

Spoerl, Charles A. The Whittaker-Henderson graduation formula A. The mixed difference case. Trans. Actuar. Soc. America 42, 292–313 (1941). [MF 5731]

The author examines a generalization of the original Whittaker-Henderson graduation formula A; in the new form the graduated series u_x is determined from the ungraduated series u_x' by minimizing the expression

$$\sum_x (u_x-u_x')^2+h\sum_x(\Delta^2u_x)^2+k\sum(\Delta^3u_x)^2,$$

where h and k are parameters. Previously [Trans. Actuar. Soc. America 38, 403–462 (1937)] the author studied a similar formula in which the expression to be minimized was $\sum(u_x-u_x')^2+h\sum(\Delta^m u_x)^2$, with m fixed. *W. Feller.*

Consael, Robert. Sur une généralisation des formules d'ajustement de E. T. Whittaker. Mitt. Verein. Schweiz. Versich.-Math. 41, 95–107 (1941). [MF 5101]

E. T. Whittaker's method of smoothing the observed values $u_1, u_2, \cdots$ of a function u_x is based on the assumptions that the deviations $u_i-\overline{u}_i$ of the observed values u_i from the true (graduated) values $\overline{u}_i$ are independent and each has a normal probability distribution with the same precision, that the "measure of roughness" $S=\sum\Delta^3\overline{u}_i$ has

a probability distribution similar to the normal law, and that the quantities involved are not correlated. In the present paper this method is extended to the case in which the quantities $u_i-\overline{u_i}$ and $\Delta^2\overline{u_i}$ are correlated.

Z. W. Birnbaum (Seattle, Wash.).

Stoner, Paul Matthew. Fitting the exponential function and the Gompertz function by the method of least squares. J. Amer. Statist. Assoc. **36**, 515–518 (1941). [MF 5883]

Ono, Katudi. Eine Ausgleichungsmethode der statistischen Reihen. Jap. J. Math. **17**, 117–126 (1940). [MF 4608]

As the author himself recognizes in an additional note, the method in question is already known [Henderson, Mathematical Theory of Graduation, chap. 4, pp. 25–46].

Buerger, M. J. Optically reciprocal gratings and their application to syntheses of Fourier series. Proc. Nat. Acad. Sci. U. S. A. **27**, 117–124 (1941). [MF 3808]

This paper is purely geometrical optics and without interest for the theory of Fourier series. *R. L. Dietzold.*

Vercelli, Francesco. Analizzatore meccanico delle curve oscillanti. Pont. Acad. Sci. Comment. **3**, 659–692 (1939). (1 plate) [MF 4106]

The author describes the construction and operation of a mechanical device for facilitating the arithmetical work involved in the analysis of a function $f(t)$ into a sum of sinusoidal functions with initially unknown, and possibly variable, amplitudes, frequencies and phases. The process appears to be laborious; but, judging from the examples given, quite good results are obtainable. *R. L. Dietzold.*

Feinstein, Lillian and Schwarzschild, Martin. Automatic integration of linear second-order differential equations by means of punched card machines. Rev. Sci. Instruments **12**, 405–408 (1941). [MF 5027]

The differential equation treated is

$$d^2y/dx^2=b(x)y+c(x),$$

to which form any second order, ordinary, linear equation can be reduced. By developing y in a Taylor series, utilizing the differential equation and neglecting terms of the fourth power or higher in ω, there results

$$y(x+\omega)+y(x-\omega)=[2+\omega^2b(x)]y(x)+\omega^2c(x).$$

The integration of the differential equation by means of the above is carried out by using the recurrence formula

$$y_{n+1}=B_ny_n-y_{n-1}+C_n,$$

where $x_n = x_0 + n\omega$, $y_n = y(x_n)$, $B_n = 2 + \omega^2b(x_n)$ and $C_n=\omega^2c(x_n)$. Similar but slightly more complicated formulas which are correct to within sixth powers of ω are also obtained. In carrying out the integration by means of punched card machines the values of B_n and C_n are computed for every x_n and are punched on cards which are later put into the feed of the multiplier punch. Five cycles of operations are carried out in arriving from y_n to y_{n+1}. These cycles comprise feeding a card and copying the values of B_n and C_n into the proper counters, multiplying these by y_n, subtracting y_{n-1}, and adding C_n, obtaining finally y_{n+1} and eliminating the card preparatory to the next step. The accuracy obtainable is estimated. The starting values of the first y and its preceding y have to be computed separately before the automatic integration is begun. *H. Poritsky.*

Sandomire, Marion M. Accumulating cubes with punch cards. J. Amer. Statist. Assoc. **36**, 507–514 (1941). [MF 5882]

McPherson, J. C. On mechanical tabulation of polynomials. Ann. Math. Statistics **12**, 317–327 (1941). [MF 5417]

The object of the paper is to show how commercial automatic accounting machines can be used to construct automatically numerical tables of mathematical functions. The lack of automatic multiplication or division prevents the direct use of these machines for this purpose. The given function $f(x)$ whose values are to be tabulated for $x=1, 2, \cdots$ is first approximated by a polynomial $P(x)$. The sequence of numbers $\{P(n)\}$ is then expressed as a sum of certain sequences $\{{}^iT_j(n)\}$, where $\{{}^0T_j(n)\}$ is a starting sequence consisting of zeros except for one number (whose value depends on $P(x)$) in the jth position, and the superscript i indicates the accumulated sum of $\{{}^0T_j(n)\}$ repeated i times. If $\{{}^iT_j(n)\}$ is fed into the machine on punched cards, the accumulated totals $\{{}^{i+1}T_j(n)\}$ can be obtained in the form of another set of punched cards. After several operations of this type, the printed values of $P(n)$ are obtained. A defect of the method is the accumulation of errors in rounding off decimals. *P. W. Ketchum.*

Crew, E. W. Calculating machines. Engineer **172**, 438–441 (1941). [MF 6072]

Wittke, Heinz. Ein vorzeichentreuer Koordinatenumformer. Allg. Vermessgs.-Nachr. **53**, 274–283, 294–301 (1941). [MF 5711]

A compound computing machine is constructed from the units of a pair of standard machines by (a) using a common power shaft to drive the separate power shafts through a pair of forward and reverse gears, thus driving both machines through the same number of revolutions but with all possible combinations of forward and backward motions and (b) mounting the two input units, one in front of the other, on a revolving carriage so that when the carriage is rotated about a horizontal axis the keyboards will be inverted and will have interchanged output units. The machine will (a) multiply or divide two numbers by a common factor in one operation, or divide one and multiply the other by the factor, etc. and (b) add to or subtract from the result in (a) the product or quotient of the original two numbers interchanged, by another common factor. Specifically, it will perform in the minimum number of operations a transformation of the form $x'=ax+by$, $y'=\pm bx\pm ay$. A number of applications to geodesy and surveying are discussed. *P. W. Ketchum* (Urbana, Ill.).

Chao, Robert F. H. Affine charts. Sci. Rep. Nat. Tsing Hua Univ. (A) **4**, 135–144 (1940). [MF 5323]

An exposition of the theory of nomographic charts. *P. W. Ketchum* (Urbana, Ill.).

MATHEMATICAL PHYSICS

Kusnetzoff, E. Über die Bestimmung des Strahlungshaushaltes in der absorbierenden und zerstreuenden Atmosphäre. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1940**, 813–842 (1940). (Russian. German summary) [MF 4077]

The author assumes that the quantities characterizing the field of radiation depend only on the frequency of radiation ν, the elevation z over the surface of the earth and the angle θ formed by the directions of a ray and of the vertical. The variable z is replaced by $\tau=\int_0^z\rho(z)dz$, where $\rho(z)$ is the density of air. The basic problem considered is that of determining the intensity $I_\nu(\tau, \theta)$ of radiation when the intensities are known at the surface of the earth and on the outer boundary of the atmosphere. The author establishes two integrodifferential equations which are eventually reduced to a single linear integral equation of Fredholm type; the latter is solved by the method of successive approximations. The author establishes a connection between the points of view of E. Gold and R. Emden. *W. J. Trjitzinsky.*

Kusnetzoff, E. Wärmeaustausch durch Strahlung in einem sich bewegenden flüssigen Mittel. Bull. Acad. Sci. URSS Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 3–28 (1941). (Russian. German summary) [MF 4273]

Die vorliegende Arbeit beschäftigt sich mit der Ableitung des vollständigen Systems der Gleichungen der Hydrodynamik und der Strahlungstheorie. *Author's summary.*

Ignatovskij, V. S. Zur Beugung an einer kreisrunden Öffnung und ebensolcher Scheibe. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 868–871 (1941). [MF 5515]

The symbol N_s or $N_s(x, \rho)$ is used to denote a function of x and ρ which satisfies the wave-equation of order s. Solutions of Maxwell's equations in cylindrical coordinates are then expressed in terms of functions N_0, N_1, N_2, where N_s, $\partial N_s/\partial\rho$, $\partial N_s/\partial x$ are continuous for $x=0$,

$$N_s=(k\rho)^s+C_sJ_s(k\rho), \quad k^{-2}\partial^2N_s/\partial x^2=-(k\rho)^s, \quad x=0;\ \rho\leqq a,$$
$$\partial N_s/\partial x=0, \quad x=0;\ \rho\geqq a.$$

With the notation $u(x, \rho)=\partial N_0/\partial x$, $u(0, \rho)=u_0(\rho^2)$, a general expression is first found for $u(x, \rho)$ in the form of a double integral and $u_0(\rho^2)$ is likewise expressed as a double integral involving $N_0(0, p)$ and its first two derivatives with respect to p. The expression for $N_0(0, p)$ as a double integral involves $u_0(p^2)$ and an integral equation is found for $u_0(p^2)$. As this equation is very complicated the author proceeds by another method in which $u_0(\rho^2)$ is expanded in a series of orthogonal functions. Much use is made of the results of former papers [cf. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 375–378, 665–667 (1939); these Rev. **1**, 350].

H. Bateman (Pasadena, Calif.).

Street, Robert E. Non-linear equations of the electromagnetic field. Proc. Nat. Acad. Sci. U. S. A. **27**, 413–417 (1941). [MF 5077]

The author writes in his introductory remarks: "The recent theory of the electromagnetic field of Born and Infeld and the earlier theory of Whittaker can be easily combined to give a formal derivation of a very general set of field equations, which contain not only these two theories as special cases but also the older theories of Maxwell and Mie." His assumptions are: (1) The usual relation between the tensor f_{kl} and the vector potential Φ_k is replaced by

$$f_{kl}=\frac{\partial\Phi_k}{\partial x^l}-\frac{\partial\Phi_l}{\partial x^k}+q_{kl},$$

where the q_{kl} are functions of f_{kl}. (2) The variation principle with a given Lagrangian l is assumed. Then the field equations are $f^{*kx}{}_{,x}=q^{*kx}{}_{,x}$, $p^{kx}{}_{,x}+m^{kx}{}_{,x}=0$, where the star denotes the dual vector and

$$p^{kl}=\frac{\partial l}{\partial f_{kl}}, \qquad m^{kl}=\tfrac{1}{2}p^{\alpha\beta}\frac{\partial q_{\alpha\beta}}{\partial f_{kl}}+\tfrac{1}{4}p^{\alpha\beta}\frac{\partial q_{\alpha\beta}}{\partial f_{\lambda\mu}}\frac{\partial q_{\lambda\mu}}{\partial f_{kl}}+\cdots.$$

The assumption $q_{kl}=0$ leads to the Born-Infeld theory; the assumptions $q_{kl}\neq0$ and that l is Maxwellian lead to the theory of Whittaker. The rest of the paper contains the conservation laws and the equations of motion.

L. Infeld (Toronto, Ont.).

Mariani, Jean. Une interprétation géométrique des équations de Maxwell dans le vide. C. R. Acad. Sci. Paris **211**, 768–771 (1940). [MF 5370]

An electromagnetic field is represented by an infinitesimal rotation $X=\sum_{i=1}^{6}e_iX_i$ in space time, which is expressible as a linear combination of three space operators X_1, X_2, X_3 (such as $y(\partial/\partial z)-z(\partial/\partial y)$) and three space-time operators X_4, X_5, X_6 (such as $x(\partial/\partial t)-t(\partial/\partial x)$). The variation of X with time in this six-space is given by $X'=S^{-1}XS=\rho X$, where $S=I+(\sum\alpha_iX_i)dt$, $\rho=I+\sigma dt$. In terms of the quantities e_i and α_i these relations become: $\sum(de_i/dt)X_i=\lim(X'-X)/dt=\sum e_i\alpha_j(X_iX_j-X_jX_i)=\sigma\sum e_iX_i$. Since the solutions of the differential equations for e_i are to be bounded at infinity, the values of σ must be pure imaginary. When the expressions $X_iX_j-X_jX_i$ are written as linear combinations of X_k, six equations in six unknowns X_k are obtained, whose determinant has six characteristic roots σ, two of them zero. Denoting by ε, $i\varphi$, α and $i\beta$ the vectors (e_1, e_2, e_3), (e_4, e_5, e_6), $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_4, \alpha_5, \alpha_6)$, the author assumes first that the lengths of ε and φ are preserved under rotation, and second that the four non-zero values of σ are equal in pairs. From this it follows that $\beta=0$, $\alpha\cdot\varepsilon=\alpha\cdot\varphi=0$, and the differential equations become $d\varepsilon/dt=\alpha\times\varepsilon$, $d\varphi/dt=\alpha\times\varphi$. The Maxwell equations are then obtained by interpreting H_x+iE_x, $\cdots$, as the Fourier integrals $\int_{-\infty}^{\infty}(k_1+i\varphi_1)e^{i\omega\cdot r}dr$, $\cdots$, and differentiating under the integral sign. *J. S. Frame* (Providence, R. I.).

Mariani, Jean. Une interprétation géométrique de l'action du champ électromagnétique. C. R. Acad. Sci. Paris **211**, 430–432 (1940). [MF 6145]

The motion of a particle of mass m and charge e in a constant electromagnetic field F_{ik} is determined by the relativistic equations

$$(1) \qquad d\xi_i/d\tau=\omega_{ik}\xi^k, \quad \xi_i=dx_i/dt; \quad i, k=1, 2, 3, 4,$$

with $\omega_{ik}(=-\omega_{ki})=eF_{ik}/mc$, where τ denotes the proper time. From the geometric point of view, the equations (1) define an infinitesimal Lorentz transformation of the vector ξ_i. The author therefore emphasizes the fact that both the changes of coordinate systems and the motions of charged

particles are described by the transformations of the same continuous group. In the case of an inhomogeneous external field these considerations remain valid in the neighborhood of a world point. *V. Bargmann* (Princeton, N. J.).

Mariani, Jean. Sur l'interprétation géométrique des équations de Dirac dans le vide. C. R. Acad. Sci. Paris **212**, 472–475 (1941). [MF 4917]

In an earlier note [cf. the preceding review] the author discussed the connection between the equations of motion of a particle and the infinitesimal transformations of a related continuous group, in particular, the connection between the equations of a relativistic mass point in an electromagnetic field and the infinitesimal Lorentz transformations. He now applies similar considerations to Dirac's equations of the electron (in momentum space) and the infinitesimal spin transformations (representation $D_{\frac{1}{2}}$ of the rotation group). *V. Bargmann.*

Mariani, Jean. Sur les équations d'onde relativistes des particules matérielles à spin quelconque en l'absence de champ. C. R. Acad. Sci. Paris **212**, 692–694 (1941). [MF 4928]

Extension of the foregoing discussion [see the preceding review] to the case of a particle with arbitrary spin [equations of Dirac [Proc. Roy. Soc. London. Ser. A. **155**, 447–459 (1936)] and Fierz [Helvetica Phys. Acta **12**, 3–37 (1939)]]. The related group is an irreducible representation $D_{jj'}$ of the Lorentz group. *V. Bargmann.*

Peng, H. W. Perturbation theory for the self-consistent field. Proc. Roy. Soc. London. Ser. A. **178**, 499–505 (1941). [MF 5243]

The Fock-Dirac equations of the self-consistent field provide one of the most accurate methods of solving approximately the n-electron problems of quantum mechanics, but, since these equations are non-linear, the ordinary perturbation theory does not apply. The author's method consists of first setting up an ordinary (linear) eigenvalue problem, whose operator is defined in terms of the n wave functions obtained by solving the unperturbed Fock-Dirac equations. The eigenfunctions of this linear problem then provide a complete orthonormal set, in terms of which the perturbed Fock-Dirac functions may be expanded, as in the usual perturbation theory. To obtain the approximate perturbed functions of a given order in this way, it is necessary to solve a system of linear algebraic equations, which the author gives explicitly. To get the approximate perturbed energy of kth order, the perturbed functions up to the $(k-1)$st order must first be found.
O. Frink (State College, Pa.).

Krishna Rao, D. The last multiplier in quantum mechanics. J. Mysore Univ. Sect. B. **2**, 1–4 (1941). [MF 5454]

Dugas has shown that in some ways the time-dependent probability density $\Psi^*\Psi$ of quantum mechanics plays the role of the last multiplier with respect to the equations (1) $dq_k/dt=i_k/\Psi^*\Psi$, taken as the equations of motion of a classical mechanical system. Here i_k is the kth component of the current vector, that is, $i_k=(h/4\pi m_k i)(\Psi^*\partial\Psi/\partial q_k - \Psi\partial\Psi^*/\partial q_k)$. The author strengthens this analogy by showing that $\Psi^*\Psi$ has another property shared by the last multipler in classical mechanics, namely, that, if A (a real function of the coordinates and time) is an integral of the equations (1), then, if Ψ is a solution of the wave equation, so is $A^1\Psi$; in other words, if $\Psi^*\Psi$ is a "last multipler," so is $A\Psi^*\Psi$. The justification for considering (1) as the equations of motion is provided by showing that the Hamilton canonical equations can be obtained from them by taking mean values. *O. Frink* (State College, Pa.).

Mathisson, Myron. Relativistic dynamics of a spinning magnetic particle. Proc. Cambridge Philos. Soc. **38**, 40–60 (1942). [MF 6034]

This is a posthumous paper left in an unfinished state and prepared for publication by P. A. M. Dirac. It is a continuation of the author's paper in Proc. Cambridge Philos. Soc. **36**, 331–350 (1940)[these Rev. **2**, 207]. The general variational method of that paper is applied to the case of a particle for which the third and higher moments may be neglected, moving in an arbitrary external field of force, to obtain the equations of motion for the angular momentum and center of mass. Then, assuming that a certain part of the energy tensor may be neglected, equations of angular momentum are derived for a charged magnetic particle moving in an arbitrary external electromagnetic field. Using these equations it is shown that, if the absolute value of the angular momentum is to satisfy an integrability condition, then the angular momentum and the electric and magnetic moments must all be parallel at any point of the world path. Equations of motion for the center of mass are next derived on the assumption that the electric moment may be neglected. It follows from these that the magnetic moment and the angular momentum are proportional, the factor of proportionality being constant along the path. Whether the electric moment must necessarily vanish is a question which was considered by the author, but which is left undecided in this paper.
O. Frink (State College, Pa.).

Bhabha, H. J. and Corben, H. C. General classical theory of spinning particles in a Maxwell field. Proc. Roy. Soc. London. Ser. A. **178**, 273–314 (1941). [MF 5167]

Dirac's theory of the radiative reactions of a point electron in classical electrodynamics [Proc. Roy. Soc. London. Ser. A. **167**, 148 (1938)] is extended to particles with dipole singularities. The equations of motion of such a particle in an external field are derived from the postulates of strict conservation of energy, momentum and angular momentum and include fully the effects of radiative reactions in the same manner as in Dirac's theory. Mass, angular momentum (that is, spin) and momentum of inertia perpendicular to the spin axis appear as arbitrary constants. In the general theory no relation exists between the electric and magnetic dipole moments of the particle and the state of its translational momentum. It is, however, possible to specialize the equations so that the dipole is either always purely magnetic or purely electric. In this case the expressions for radiative reaction simplify considerably. As an application of the theory the scattering of light by such a particle is worked out. It is pointed out that a classical dipole as introduced in this paper does not describe the ordinary electronic spin which is a pure quantum effect.
L. W. Nordheim (Durham, N. C.).

Bhabha, H. J. General classical theory of spinning particles in a meson field. Proc. Roy. Soc. London. Ser. A. **178**, 314–350 (1941). [MF 5168]

The work of the paper reviewed above is extended to include a finite rest mass for the field particles so as to be

applicable to the meson field in interaction with neutrons and protons. The scattering of mesons by nuclear particles is treated under the assumption of a pure dipole (mesomagnetic) coupling and the following results are obtained. The scattering cross section for transversely polarized mesons is due mainly to the interaction with the nuclear spin coordinate. The calculated cross section near the maximum is much higher than the observed one. For longitudinally polarized mesons the scattering is due only to interaction with the translational and not with the rotational motion of the neutrons or protons. The classical theory is, however, applicable only to electrically neutral mesons. For an extension to charged mesons it would be necessary to assume that the nuclear particles may exist also in different states of integral charge (for instance, $-e$ and $2e$), in which case the above result could be taken over without much change. *L. W. Nordheim* (Durham, N. C.).

Petiau, Gérard. Sur la théorie du spin. C. R. Acad. Sci. Paris 211, 765–767 (1940). [MF 5369]

The spin matrices of a particle of spin $nh/4\pi$ are taken to be

$$S_i=-\frac{1}{2}\frac{h}{2\pi}\sum_p^n \sigma_i^{(p)}=-\frac{n}{2}\frac{h}{2\pi}X_i,$$

where $\sigma_i^{(p)}$ are the three spin matrices satisfying

$$\sigma_i^{(p)}\sigma_j^{(p)}=-\sigma_j^{(p)}\sigma_i^{(p)}=i\sigma_k^{(p)},$$

$$\sigma_i^{(p)}\sigma_i^{(p)}=1,\quad \sigma_i^{(p)}\sigma_j^{(q)}=\sigma_j^{(q)}\sigma_i^{(p)}.$$

The author then indicates a method for deriving the characteristic equations of the matrices X_i. For small values of n these equations are given explicitly. It is stated that these equations lead to the following result: if n is even the proper values of S_i are $\pm kh/2\pi$ ($k=0, 1, 2, \cdots, n/2$); if n is odd the proper values of S_i are $\pm(k+\frac{1}{2})h/2\pi$ ($k=0, 1, \cdots, (n-1)/2$). *A. H. Taub.*

Petiau, Gérard. Sur la théorie du corpuscule de spin 2. C. R. Acad. Sci. Paris 212, 292–295 (1941). [MF 4899]

Let (C) be a particle described by the wave equations

$$(1)\quad \sum_\mu \partial_\mu(\Gamma_\mu)_{il}\psi_l=\lambda\psi_i,\ \mu=1,\cdots,4;\ x_4=ict;\ \lambda=(2\pi/h)m_0c.$$

The particle $(2C)$, then, is defined by the equations

$$(2)\quad \sum_\mu \partial_\mu(\Gamma_\mu)_{il}\psi_{lk}=\lambda\psi_{ik};\quad \sum_\mu \partial_\mu(\Gamma_\mu)_{km}\psi_{im}=\lambda\psi_{ik}.$$

Starting from de Broglie's equations for the photon the author derives the equations (2) for the corresponding particle $(2C)$ of spin 2. They can be decomposed into six separate systems: one system containing 35 linearly independent quantities [it is related to M. Fierz's equations for a particle of spin 2 [Helvetica Phys. Acta 12, 3–37 (1939)]], two systems containing 15 independent quantities, and three systems containing 35 independent quantities.

V. Bargmann (Princeton, N. J.).

Fierz, Markus. Klassische Theorie der Streuung geladener Mesonen. Helvetica Phys. Acta 14, 257–270 (1941).

It is shown that it is possible to develop a classical field theory for the interaction of mesons with nuclear particles. For this it is necessary to introduce a charge coordinate for the nuclear particle, which, in classical theory, can vary continuously. The equations of motion for this coordinate are obtained from the laws of conservation of energy, momentum and charge in a manner analogous to Dirac's classical theory of point electrons [Proc. Roy. Soc. London. Ser. A. 167, 148 (1938)]. They include fully the effects of radiative reactions. The theory can be made to correspond to the two special cases: (I) The nuclear particle is capable of only two charge states (proton and neutron). (II) The nuclear particle can assume all positive and negative integer charge states. The scattering of longitudinally polarized mesons is treated. The results obtained show a close correspondence to results obtained by quantum theory.

L. W. Nordheim (Durham, N. C.).

Sarginson, Kathleen. Reciprocity. VI. The wave function of the meson. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 77–92 (1941). [MF 5631]

The treatment of the equations of the meson is like that adopted for the Dirac equation by K. Fuchs [Proc. Roy. Soc. Edinburgh 60, 147–163 (1940); these Rev. 2, 143]. The "classical" meson equations are

$$\partial\phi_\beta/\partial x^\alpha-\partial\phi_\alpha/\partial x^\beta=\epsilon_1' X_{\alpha\beta},\quad \partial X_{\alpha\beta}/\partial x^\alpha=\epsilon_2'\phi_\beta,$$

where $\epsilon_1'=\epsilon_2'=m_0c/h$, and m_0 is the rest mass. With x^0, x^1, x^2, $x^3=ct$, x, y, z, $x^m=g^{mn}x_n$, $p_n=-ih\partial/\partial x^n$, $g^{mn}=0$, $m\neq n$, $g^{nn}=1$, $n\neq 0$, $g^{00}=-1$, the wave equation of the meson is taken in Kemmer's form

$$(\beta^k p_k-iP\beta)\psi^P=0,$$

where the β^k are matrices satisfying the commutation relations

$$\beta^k\beta^m\beta^n+\beta^n\beta^m\beta^k=g^{km}\beta^n+g^{nm}\beta^k$$

and β is a diagonal matrix chosen so that each component of ψ^P will satisfy the equation $(p_kp^k-P^2)\psi=0$, which, for imaginary values of P, is the Klein-Gordon equation. A transformation to polar coordinates is given and the equations holding for the adjoint $\psi^\dagger$ of ψ are given for imaginary values of P. The spin operators, their commutation relations and their representation are next given. As usual vectors ${}^sM$, ${}^sN$ are introduced, also the invariant ${}^sK^2={}^sM^2-{}^sN^2$, and proper values are found for ${}^sM_2$, ${}^sM^2$, ${}^sK^2$. A solution is given of the wave equation and finally an expression is obtained in polar coordinates for the wave equation in p-space and two independent solutions are written down.

H. Bateman (Pasadena, Calif.).

Karnovsky, M. I. Interaction acoustical impedance of spherical radiators and resonators. C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, 40–43 (1941). [MF 5851]

Let A and B be two spheres of radius r_0 distant d apart. Let the oscillation velocities be $v_i=V_i\exp j(\omega t-\psi_i)$, $i=1, 2$; $\psi_1=0$. The pressure $p_{21}(t)$ on B induced by the oscillation of A is $j\omega\rho\phi_1|_d$, where ϕ_1 is the velocity potential due to A and is given by the classical formula. Then the mean induced power from B is essentially

$$T^{-1}\int_0^T p_{21}'\pi r_0^2 V_2'(t)dt=\tfrac{1}{2}V^2R_{21}$$

if $V_2V_1=V_2$, where R_{21} plays the role of a resistance and a prime denotes the real part. This expression is evaluated for $\psi_2=0$ and $\psi_2=-\pi/2$ and yields in phase, or resistance, and out of phase, or reactive, components. These components or parameters give convenient specifications for the radiation characteristics. The parameters and the power output are evaluated for nearby point radiators A and B and the combination of a point radiator A and a nearby point resonator B, that is to say, for the case $kr_0\ll 1$, $kd\ll 1$, where k is inversely proportional to the velocity of wave propagation. [Practically all of the results to this point are obvious implications of the classical Hertzian doublet computations

except for the use of "sound" terminology rather than that of electrodynamics.] It is indicated that the power output at large distances from a radiator may be increased or diminished by a nearby resonator depending on the internal dissipation resistances of the resonator. *D. G. Bourgin.*

Pipes, Louis A. An operational treatment of electromagnetic waves along wires. J. Appl. Phys. 12, 800–810 (1941).

The well-known equations of electromagnetic waves guided by two parallel wires are derived via the Laplacian transformation. The rigorous treatment of the non-dissipative case is followed by that involving small dissipation, represented as perturbation of the former. This yields the damping and distortion of the front of an initially square wave due to skin effect dispersion, by means of the corresponding convergent operational expansion, whose first fractional exponent term is taken into account. Numerical conclusions based on this term alone concerning the maximum slope of transient are not sufficiently substantiated, as it is not clear whether at the time of steepest ascent the operational "image" of the first term is still a good approximation of the rigorous solution.

H. G. Baerwald (Cleveland, Ohio).

Cherry, E. C. and Rivlin, R. S. Non-linear distortion, with particular reference to the theory of frequency modulated waves. I. Philos. Mag. (7) 32, 265–281 (1941). [MF 5911]

Explicit expressions are derived for the modulation products in the output current of a non-linear circuit element, resulting from the application of an electromotive force consisting of any finite number of sinusoidal components. This analysis is carried out by traditional steady-state methods, as opposed to the more recondite mathematical treatment of Carson and Fry [Bell System Tech. J. 16, 513–540 (1937)]. *R. M. Foster* (New York, N. Y.).

de-Stefano, Alberto. Sui teoremi di reciprocità nella radiotelegrafia. Nuovo Cimento (N.S.) 17, 175–182 (1940). [MF 5609]

It is shown that the reciprocity theorems of Carson and Ballantine are actually identical. A relation of the Carson-Ballantine type is derived which is not restricted to sinusoidal impressed fields, but is valid for rather general phenomena. *H. G. Baerwald* (Cleveland, Ohio.)

Boni, Alessandro. Teoria non lineare dei circuiti elettrici. Atti del secondo Congresso dell'Unione Matematica Italiana, Bologna, 4–6 Aprile 1940, pp. 453–462. [MF 5762]

The fundamentals and methods of non-linear circuit theory are briefly reviewed: equations of the fundamental cases; application of perturbation methods to quasi-linear cases; stability at large and "cycles limites"; graphical and numerical integration methods. Some selected references are listed [none of the important papers of the Moscow school are quoted]. An original contribution is given: a numerical integration method for the auto-oscillation equation with one degree of freedom; the associated integral equation is given and attacked by the step-by-step method involving a pair of recurrence relations. As demonstrated by a numerical example, this is superior to conventional methods [Cauchy-Lipschitz, Isoclinics]. *H. G. Baerwald.*

Clark, C. E. On the existence of electrical networks. Bull. Amer. Math. Soc. 47, 769–771 (1941). [MF 5485]

The author studies conditions for the existence of electrical networks which possess a set of circuits with preassigned resistances in common, that is, a set of circuits satisfying a given intersection matrix. The networks found, however, include a multiplicity of other independent circuits in addition to those entering the given matrix; hence this intersection matrix is not the ordinary mesh matrix of electrical theory for the network obtained in this paper, but only a part of the complete mesh matrix. Thus the theorems derived are of extremely limited application.

R. M. Foster (New York, N. Y.).

Chow, Wei-Liang. On electric networks. J. Chinese Math. Soc. 2, 321–339 (1940). [MF 4217]

This paper gives the extension to unrestricted networks of a rule attributed to K. T. Wang [Nat. Res. Inst. Engrg., Acad. Sinica, Mem. no. 2] for evaluating various determinants useful in electrical network theory by writing down the expanded form of the determinants directly from the configuration of the network. In many respects this paper is a continuation of the work of S. L. Ting [Chinese J. Phys. 1, no. 3, 18–40 (1935)] and C.-T. Tsai [Chinese J. Phys. 3, 148–181 (1939)] in extending the scope of the original proposal of Wang. Wang's rule is essentially a development of Kirchhoff's original rule of 1847 into a more readily applied form. *R. M. Foster* (New York, N. Y.).

Cooper, M. D. and Burington, Richard S. On transient similarity and equivalence in n-mesh linear networks. Philos. Mag. (7) 32, 302–314 (1941). [MF 5913]

The theory of elementary divisors of λ-matrices is applied to the classification of linear passive electrical networks according to their transient behavior. Two linear networks of n meshes are called "transiently similar" if their λ-matrices $A=(a_{rs})$, $a_{rs}=L_{rs}\lambda^2+R_{rs}\lambda+S_{rs}$; $\lambda=d/dt$; $L, R, S=$(real, positive semidefinite) inductance, resistance, electrical stiffness matrices, respectively, have equal Segre characteristics (numbers and multiplicities of elementary divisors in the Smith normal form); they are called "transiently similar" if, in addition, their divisors are the same, that is, if A_1 and A_2 are equivalent. The Segre characteristics are here based on the field of real numbers; explicit distinction is thus made between real and complex invariant factors, a periodic or oscillatory character of the associated normal oscillations. The complete classification table of two-mesh networks is given, in re-representation of W. Quade's solution of the same problem [Klassifikation der Schwingungsvorgänge in gekoppelten Stromkreisen, Leipzig, 1933] except for the distinction just mentioned; also, those subclasses which have passive "representatives," but are not associated with physically reliable networks without ideal transformers, are singled out. For complete classification for $n=3$ the M. S. Thesis of one of the authors is quoted. Electrical networks with m independent terminal pairs and different number of meshes $n \geqq m$ may be steady-state (s.s.) equivalent, as this requires equivalence under m non-singular affine transformations only. Thus s.s. equivalence does not imply transient (t.) equivalence as defined here (but the opposite is true); differences in such normal oscillations as have zero amplitudes in the meshes containing the terminal pairs is compatible with s.s. but not with t. equivalence; a corresponding numerical example is given. *H. G. Baerwald* (Cleveland, Ohio).